49. *On a Problem of Dinaburg and Sinai*

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 14, 1992)

§1. **Introduction.** Let N be a sufficiently large integer. Let

$$F_N = \{a/b\,;\, 1 \leq a \leq b \leq N,\ (a,\, b) = 1,\ a \text{ and } b \text{ are integers}\}.$$

For any fraction a/b in F_N, we can associate the minimum positive integer $x_0 \leq b$ such that

$$|\,ax_0 - by_0\,| = 1$$

for some integer $y_0 \geq 1$. Let α_1, β_1, α_2 and β_2 be real numbers satisfying

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1.$$

Then we put

$$S_N = \{a/b \in F_N\,;\, \alpha_1 N < a < \beta_1 N < \alpha_2 N < b < \beta_2 N\}.$$

Dinaburg and Sinai [1] have studied the distribution of

$$x_0/b$$

as a/b belongs to S_N and $N \to \infty$. We shall improve both their results and Remark by Voronin and Tvnek in p.171 of [1].

For any a/b in F_N, we may associate the minimum positive integer $x_1 \leq b$ such that

$$ax_1 - by_1 = 1$$

for some integer $y_1 \geq 1$. We may also treat the distribution of

$$x_1/b$$

as a/b belongs to F_N or S_N and $N \to \infty$.

We may describe x_0/b in two ways. For $(a,\, b) = 1$, let $\bar{a}$ be the unique positive integer $\leq b$ such that $a\bar{a} \equiv 1 (mod\ b)$. By the definition of x_0, we see first that

$$x_0 = Min(\bar{a},\, b - \bar{a}).$$

Wenext express x_0/b in terms of the continued fraction expansion of a/b. We denote

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$$

by $[a_1, a_2, \ldots, a_n]$ and also by p_n/q_n for $n \geq 1$, where $a_1, a_2, \ldots$ and a_n are positive integers. We define $p_0 = 0$ and $q_0 = 1$. Now suppose that

$$a/b = [a_1,\, a_2, \ldots,\, a_s]$$

with the minimum integer $s \geq 1$. Thus we suppose that $a_s \geq 2$ unless $a/b = 1$. When s is odd, then $p_s q_{s-1} - q_s p_{s-1} = (-1)^{s+1}$ with $p_s = a$, $q_s = b$ and $q_{s-1} = \bar{a}$. Thus

$$x_0/b = \bar{a}/b = q_{s-1}/q_s = [a_s,\, a_{s-1}, \ldots,\, a_2,\, a_1].$$

When s is even, then $p_s = a$, $q_s = b$ and $q_{s-1} = b - \bar{a}$. Thus in this case we also have

$$x_0/b=(b-\bar{a})/b=q_{s-1}/q_s=[a_s, a_{s-1},\dots, a_2, a_1].$$

We may notice that for $b\geq 3$, $\bar{a}<b-\bar{a}$ if and only if s is odd.

Similarly, we see first that $x_1=\bar{a}$. If the length s of the continued fraction expansion of a/b is odd, then $q_{s-1}=\bar{a}$ and $x_1/b=q_{s-1}/q_s=[a_s, a_{s-1}, \dots, a_2, a_1]$. If s is even, then $q_{s-1}=b-\bar{a}$ and

$$\begin{aligned}x_1/b=\bar{a}/b=1-(q_{s-1}/q_s)&=1-[a_s, a_{s-1},\dots, a_2, a_1]\\&=[1, a_s-1, a_{s-1}, a_{s-2},\dots, a_2, a_1].\end{aligned}$$

Namely, we have $x_1/b=[a'_t, a'_{t-1}, \dots, a'_2, a'_1]$ if $a/b=[a'_1, a'_2, \dots, a'_t]$ with the odd integer $t\geq 1$.

Dinaburg and Sinai [1] have reduced their problem to the question of whether a certain special flow over the natural extension of the Gauss transformation in the theory of continued fraction is mixing (cf. p. 165 of [1]). Our approach is elementary and we shall use the estimate of Kloosterman sums as is also noticed in Remarks in p. 171 of [1].

§2. Some lemmas. We start with noticing the following lemma which says that a/b in F_N is uniformly distributed.

Lemma 1. *For a given B in $0\leq B<1$,*

$$\sum_{a/b\in F_N, B\leq a/b<B+x}\cdot 1=x\frac{N^2}{2\zeta(2)}+O(N\log N)$$

uniformly for x in $0\leq x\leq 1-B$, where $1/\zeta(2)=6/\pi^2$.

Proof. The left hand side is

$$=\sum_{d\leq N}\mu(d)\sum_{d\leq N,\ d|b}\ \sum_{d|a, bB\leq a<b(B+x)}\cdot 1=\sum_{d\leq N}\mu(d)\sum_{b\leq N/d}\ \sum_{bB\leq a<b(B+x)}\cdot 1$$

$$=\sum_{d\leq N}\mu(d)\sum_{b\leq N/d}(xb+O(1))=x\frac{N^2}{2\zeta(2)}+O(N\log N).$$

We next treat the same problem for the fractions in S_N. By the definition, a/b in S_N must satisfy $A<a/b<A+\Delta$, where we put $A=\alpha_1/\beta_2$ and $\Delta=\beta_1/\alpha_2-\alpha_1/\beta_2$. We shall prove in the following lemma that a/b is not uniformly distributed in the interval $(A, A+\Delta)$.

Lemma 2. *For any x in $0\leq x\leq\Delta$, we have*

$$\sum_{a/b\in S_N,\ A<a/b<A+x}\cdot 1=g(x)\frac{N^2}{\zeta(2)}(\beta_2-\alpha_2)(\beta_1-\alpha_1)+O(N\log N),$$

where $g(x)$ will be defined below.

We define $g(x)$ in the following three cases, separately.

Case I. $\beta_1/\beta_2<\alpha_1/\alpha_2$.

$$g(x)=\begin{cases}g_1(x) & \text{for } \Delta_1<x\leq\Delta\\ \dfrac{1}{\beta_2-\alpha_2}\Big(\beta_2-\dfrac{1}{2}\dfrac{1}{A+x}(\beta_1+\alpha_1)\Big) & \text{for } \Delta_2<x\leq\Delta_1\\ g_2(x) & \text{for } 0\leq x\leq\Delta_2,\end{cases}$$

where we put $\Delta_1=\alpha_1/\alpha_2-A$, $\Delta_2=\beta_1/\beta_2-A$,

$$g_1(x)=\frac{1}{(\beta_2-\alpha_2)(\beta_1-\alpha_1)}\Big(\alpha_1\alpha_2-\frac{1}{2}(A+x)\alpha_2^2$$

$$-\frac{1}{2}\frac{\beta_1^2}{A+x}+(\beta_1-\alpha_1)\beta_2\Big)$$

and

$$g_2(x)=\frac{1}{(\beta_2-\alpha_2)(\beta_1-\alpha_1)}\Big(\frac{(A+x)\beta_2^2}{2}+\frac{\alpha_1^2}{2(A+x)}-\alpha_1\beta_2\Big).$$

Case II. $\beta_1/\beta_2>\alpha_1/\alpha_2$.

$$g(x)=\begin{cases} g_1(x) & \text{for } \Delta_2<x\leq\Delta \\ \dfrac{1}{\beta_1-\alpha_1}\Big(\dfrac{1}{2}(A+x)(\beta_2+\alpha_2)-\alpha_1\Big) & \text{for } \Delta_1<x\leq\Delta_2 \\ g_2(x) & \text{for } 0\leq x\leq\Delta_1. \end{cases}$$

Case III. $\beta_1/\beta_2=\alpha_1/\alpha_2$.

$$g(x)=\begin{cases} g_1(x) & \text{for } \Delta_1=\Delta_2<x\leq\Delta \\ g_2(x) & \text{for } 0\leq x\leq\Delta_1=\Delta_2. \end{cases}$$

In any case, we have $g(\Delta)=1$ and $g(0)=0$.

Proof of Lemma 2. In the Case I, we have $\Delta_2<\Delta_1<\Delta$. We shall treat only for $\Delta_1<x\leq\Delta$ in this case, since the rests are similar.

$$\sum_{a/b\in S_N,\ A<a/b<A+x}\cdot 1=\sum_{\alpha_2N<b<\beta_2N}\ \sum_{\substack{\alpha_1N<a<\beta_1N\\ bA<a<b(A+x)\\ (a,b)=1}}\cdot 1$$

$$=\sum_{\substack{\alpha_2N<b<\beta_2N\\ \beta_1N/(A+x)<b}}\ \sum_{\substack{\alpha_1N<a<\beta_1N\\ (a,b)=1}}\cdot 1+\sum_{\substack{\alpha_2N<b<\beta_2N\\ \alpha_1N/(A+x)<b\leq\beta_1N/(A+x)}}\ \sum_{\substack{\alpha_1N<a<b(A+x)\\ (a,b)=1}}\cdot 1$$

$$=S_1+S_2,\ \text{say}.$$

$$S_1=\sum_{d\leq N}\mu(d)\sum_{\beta_1N/d(A+x)<b<\beta_2N/d}\ \sum_{\alpha_1N/d<a<\beta_1N/d}\cdot 1$$

$$=N^2(\beta_1-\alpha_1)\Big(\beta_2-\frac{\beta_1}{A+x}\Big)\frac{1}{\zeta(2)}+O(N\log N).$$

We have also

$$S_2=\sum_{d\leq N}\mu(d)\sum_{\alpha_2N/d<b\leq\beta_1N/d(A+x)}\Big(b(A+x)-\frac{\alpha_1N}{d}\Big)+O(N\log N)$$

$$=N^2\frac{1}{\zeta(2)}\Big\{\frac{\beta_1^2}{2(A+x)}-\frac{1}{2}(A+x)\alpha_2^2-\frac{\alpha_1\beta_1}{A+x}$$

$$+\alpha_1\alpha_2\Big\}+O(N\log N).$$

These give our result for the present case.

§3. The distribution of x_1/b. We recall that $x_1=\bar{a}$. For any $0\leq B<1$, $0\leq x\leq 1-B$ and any $0\leq y\leq 1$, we put

$$F_N(B,x,y)=\{a/b\in F_N;B\leq a/b<B+x,\ \bar{a}/b<y\}.$$

Similarly, we define, for $0\leq x\leq\Delta$ and $0\leq y\leq 1$,

$$S_N(x,y)=\{a/b\in S_N;A<a/b<A+x,\ \bar{a}/b<y\},$$

where A and Δ are the same as in the previous section. We shall evaluate the cardinalities $f_N(B,x,y)$ and $s_N(x,y)$ of $F_N(B,x,y)$ and $S_N(x,y)$, respectively. The following theorems will be proved. ε denotes always an arbitralily small positive number.

Theorem 1. *For any* $0 \leq B < 1$, $0 \leq x \lesssim 1 - B$ *and* $0 \leq y \leq 1$, *we have*

$$f_N(B, x, y) = yxN^2/2\zeta(2) + O(N^{3/2+\varepsilon}).$$

Theorem 2. *For any* $0 \leq x \leq \Delta$ *and* $0 \leq y \leq 1$, *we have*

$$s_N(x, y) = yg(x)\, N^2/\zeta(2)\, (\beta_2 - \alpha_2)(\beta_1 - \alpha_1) + O(N^{3/2+\varepsilon}),$$

where $g(x)$ *is the same as in Lemma* 2.

As a special case of Theorem 2, we get the following corollary.

Corollary 1. *For any* y *in* $0 \leq y \leq 1$, *we have*

$$\left| \frac{\#\{a/b \in S_N\,;\, 0 < x_1/b < y\}}{\# S_N} - y \right| \ll N^{-\frac{1}{2}+\varepsilon},$$

where $\# S$ *denotes the cardinality of the set* S.

We shall prove only Theorem 2, since Theorem 1 can be proved in a similar manner.

Proof of Theorem 2. Let $\chi_I(t)$ be the characteristic function of the interval I. Let δ be a number in $0 < \delta < 1/4$. Suppose first that $2\delta \leq x \leq 1 - 2\delta$ and $2\delta \leq y \leq 1 - 2\delta$. Then by Vinogradov's Lemma 2 in p. 196 of [3], we get two periodic functions $\Psi_1(t)$ and $\psi_1(t)$ of period 1 such that

(i) $\Psi_1(t) - \chi_{[A,A+x)}(t) = 0$ except in $(A-\delta, A) \cup (A+x, A+x+\delta)$,

$\psi_1(t) - \chi_{[A,A+x)}(t) = 0$ except in $(A, A+\delta) \cup (A+x-\delta, A+x)$,

$0 < \Psi_1(t) < 1$ for any t in $(A-\delta, A) \cup (A+x, A+x+\delta)$

and

$0 < \psi_1(t) < 1$ for any t in $(A, A+\delta) \cup (A+x-\delta, A+x)$

and

(ii) $\Psi_1(t) = x + \sum_{m=1}^{\infty} (a_m e(tm) + b_m e(-tm))$

and

$\psi_1(t) = x + \sum_{m=1}^{\infty} (a'_m e(tm) + b'_m e(-tm))$,

where $e(t) = e^{2\pi it}$ and

$$|a_m|, |b_m|, |a'_m|, |b'_m| < Min\left(\frac{1}{\pi m}, x, \frac{1}{(\pi m)^2\delta}\right).$$

Similarly, for the interval $[0, y)$ we get two functions $\Psi_2(t)$ and $\psi_2(t)$ having the same properties as above with the Fourier coefficients c_m, d_m, c'_m and d'_m, respectively.

Using these functions, we have

$$\Sigma_2 \equiv \sum_{a/b\in S_N} \psi_1\left(\frac{a}{b}\right) \psi_2\left(\frac{\bar{a}}{b}\right) \leq \sum_{\substack{a/b\in S_N \\ A<a/b<A+x \\ \bar{a}/b<y}} \cdot 1 \leq \sum_{a/b\in S_N} \Psi_1\left(\frac{a}{b}\right)\Psi_2\left(\frac{\bar{a}}{b}\right) \equiv \Sigma_1, \text{ say.}$$

We shall treat only Σ_1.

$$\Sigma_1 = y \sum_{a/b\in S_N} \Psi_1\left(\frac{a}{b}\right) + \sum_{a/b\in S_N} \Psi_1\left(\frac{a}{b}\right)\left(\Psi_2\left(\frac{\bar{a}}{b}\right) - y\right)$$

$$= y \sum_{a/b\in S_N} \chi_{[A,A+x)}\left(\frac{a}{b}\right) + y \sum_{a/b\in S_N} \left(\Psi_1\left(\frac{a}{b}\right) - \chi_{[A,A+x)}\left(\frac{a}{b}\right)\right)$$

$$+ x \sum_{a/b \in S_N} \left(\Psi_2\left(\frac{\bar{a}}{b}\right) - y\right) + \sum_{a/b \in S_N} \left(\Psi_1\left(\frac{a}{b}\right) - x\right)\left(\Psi_2\left(\frac{\bar{a}}{b}\right) - y\right)$$

$$= \sum_3 + \sum_4 + \sum_5 + \sum_6, \text{ say.}$$

By Lemma 2, we get

$$\sum_3 = yg(x)(\beta_2 - \alpha_2)(\beta_1 - \alpha_1)\frac{N^2}{\zeta(2)} + O(N \log N).$$

$$\sum_4 \ll \sum_{\substack{a/b \in S_N \\ A-\delta \le a/b \le A}} \cdot 1 + \sum_{\substack{a/b \in S_N \\ A+x \le a/b \le A+x+\delta}} \cdot 1 \ll \sum_{\substack{a/b \in F_N \\ A-\delta \le a/b \le A}} \cdot 1 + \sum_{\substack{a/b \in F_N \\ A+x \le a/b \le A+x+\delta}} \cdot 1.$$

Applying Lemma 1 to the last two sums, we get

$$\sum_4 \ll N^2\delta + N \log N.$$

We take $H = N$ and $\delta = 1/\sqrt{N}$ below. By the definition of $\Psi_2(t)$, we get

$$\sum_5 = x \sum_{a/b \in S_N} \sum_{1 \le m \le H} \left(c_m e\left(\frac{\bar{a}}{b} m\right) + d_m e\left(-\frac{\bar{a}}{b} m\right)\right) + O\left(\frac{N^2}{H\delta}\right)$$

$$\ll \sum_{1 \le m \le H} \frac{1}{m} \left| \sum_{a/b \in S_N} e\left(\frac{\bar{a}}{b} m\right) \right| + \frac{N^2}{H\delta}.$$

Using the estimates on Kloosterman sums (cf. Lemma 4 in p. 36 of Hooley [2]), the last inner sum is

$$= \sum_{\alpha_2 N < b < \beta_2 N} \sum_{\substack{\alpha_1 N < a < \beta_1 N \\ (a,b)=1}} e\left(\frac{\bar{a}}{b} m\right) \ll \sum_{\alpha_2 N < b < \beta_2 N} b^{\frac{1}{2}+\varepsilon}(b, m)^{\frac{1}{2}},$$

where (b, m) is the greatest common divisor of b and m. Thus we get

$$\sum_5 \ll \sum_{1 \le m \le H} \frac{1}{m} \sum_{\alpha_2 N < b < \beta_2 N} b^{\frac{1}{2}+\varepsilon}(b, m)^{\frac{1}{2}} + \frac{N^2}{H\delta} \ll N^{\frac{3}{2}+\varepsilon}.$$

We shall finally treat $\sum_6$.

$$\sum_6 = \sum_{a/b \in S_N} \left(\sum_{1 \le j \le H} \left(a_j e\left(\frac{a}{b} j\right) + b_j e\left(-\frac{a}{b} j\right)\right) + O\left(\frac{1}{H\delta}\right)\right)$$

$$\cdot \left(\sum_{1 \le m \le H} \left(c_m e\left(\frac{\bar{a}}{b} m\right) + d_m e\left(-\frac{\bar{a}}{b} m\right)\right) + O\left(\frac{1}{H\delta}\right)\right)$$

$$\ll \sum_{1 \le j, m \le H} \frac{1}{jm} \left| \sum_{a/b \in S_N} e\left(\frac{a}{b} j + \frac{\bar{a}}{b} m\right) \right|$$

$$+ \sum_{1 \le j, m \le H} \frac{1}{jm} \left| \sum_{a/b \in S_N} e\left(-\frac{a}{b} j + \frac{\bar{a}}{b} m\right) \right|$$

$$+ O\left(\frac{\log H}{H\delta} N^2\right) + O\left(\frac{N^2}{H^2\delta^2}\right).$$

Estimating the last two inner sums by Lemma 4 of Hooley [2], we get

$$\sum_6 \ll \sum_{1 \le j, m \le H} \frac{1}{jm} \sum_{\alpha_2 N < b < \beta_2 N} b^{\frac{1}{2}+\varepsilon}(b, m)^{\frac{1}{2}} + \frac{N^2 \log N}{H\delta} + \frac{N^2}{H^2\delta^2} \ll N^{\frac{3}{2}+\varepsilon}.$$

Thus we have obtained

$$\sum_1 = yg(x)(\beta_2 - \alpha_2)(\beta_1 - \alpha_1)\frac{N^2}{\zeta(2)} + O(N^{\frac{3}{2}+\varepsilon}).$$

Similarly, we get

$$\sum\nolimits_2 = yg(x)(\beta_2 - \alpha_2)(\beta_1 - \alpha_1)\frac{N^2}{\zeta(2)} + O(N^{\frac{3}{2}+\varepsilon}).$$

Hence, we get

$$s_N(x, y) = yg(x)(\beta_2 - \alpha_2)(\beta_1 - \alpha_1)\frac{N^2}{\zeta(2)} + O(N^{\frac{3}{2}+\varepsilon}).$$

Similarly, we can treat the case when either $2\delta \leq x \leq 1 - 2\delta$ or $2\delta \leq y \leq 1 - 2\delta$ fails (cf. p. 244 of Vinogradov [3]).

§3. The distribution of x_0/b. We recall that

$$x_0 = \begin{cases} \bar{a} & \text{if } \bar{a} \leq b/2 \\ b - \bar{a} & \text{if } \bar{a} \gtrsim b/2, \end{cases}$$

Thus $0 < x_0/b \leq 1/2$. As a consequence of Theorem 2, we see the following.

Theorem 3. *For any x in $0 \leq x \leq \Delta$ and y in $0 < y \leq 1/2$, we have*

$$u_N(x, y) = 2yg(x)(\beta_2 - \alpha_2)(\beta_1 - \alpha_1)\frac{N^2}{\zeta(2)} + O(N^{\frac{3}{2}+\varepsilon}),$$

where $u_N(x, y)$ is the cardinality of the set

$$U_N(x, y) = \{a/b \in S_N\,;\, A < a/b < A + x \text{ and } 0 < x_0/b < y\}.$$

To see this, we notice only that

$$\begin{aligned} u_N(x, y) &= \sum_{\alpha_2 N < b < \beta_2 N} \sum_{\substack{\alpha_1 N < a < \beta_1 N, (a,\ b)=1 \\ A < a/b < A+x,\ \bar{a} \leq b/2,\ \bar{a}/b < y}} \cdot 1 \\ &\quad + \sum_{\alpha_2 N < b < \beta_2 N} \sum_{\substack{\alpha_1 N < a < \beta_1 N,\ (a,b)=1 \\ A < a/b < A+x, b/2 < \bar{a} < b,\ 1 - \bar{a}/b < y}} \cdot 1 \\ &= \sum_{\alpha_2 N < b < \beta_2 N} \sum_{\substack{\alpha_1 N < a < \beta_1 N,\ (a,b)=1 \\ A < a/b < A+x,\ \bar{a}/b < y}} \cdot 1 \\ &\quad + \sum_{\alpha_2 N < b < \beta_2 N} \sum_{\substack{\alpha_1 N < a < \beta_1 N,\ (a,b)=1 \\ A < a/b < A+x,\ 1-y < \bar{a}/b < 1}} \cdot 1. \end{aligned}$$

At this stage we use Theorem 2 and get Theorem 3 as described above.

As a special case of this theorem, we get the following corollary.

Corollary 2. *For any y in $0 < y \leq 1/2$, we get*

$$\left| \frac{\#\{a/b \in S_N\,;\, 0 < x_0/b < y\}}{\# S_N} - 2y \right| \ll N^{-\frac{1}{2}+\varepsilon}.$$

This should be compared with Cor. 1 in the previous section and also with Dinaburg and Sinai's theorem.

References

[1] E. I. Dinaburg and Y. G. Sinai: Statistics of the solutions of the integral equation $ax - by = \pm 1$. Funct. Anal. Appl., **24**, 165–171 (1990).

[2] C. Hooley: Applications of Sieve Methods to the Theory of Numbers. Cambridge Univ. Press (1976).

[3] I. M. Vinogradov: Selected Works. Springer-Verlag (1985).