28. *Retractive Nil-extensions of Regular Semigroups. I*

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Abstract: As retract extensions can be more easily constructed than many other kinds of extensions, it is of interest to know whether a given extension is a retract extension. The purpose of this paper is to give some criterions for retractive nil-extensions of semigroups, especially for the very important class of regular semigroups.

As retract extensions can be more easily constructed than many other kinds of extensions, it is of interest to know whether a given extension is a retract extension. It is known (see [5, p. 89]) that *every retract extension of a semigroup by a semigroup with zero is a subdirect product of these semigroups.* The converse of this assertion is given, here, by Lemma 1. This result is crucial in the proof of the main result of this paper which is given by Theorem 1: *A semigroup S is a retractive nil-extension of a regular semigroup K iff S is a subdirect product of K and a nil-semigroup.* For the related results see [2] and [3].

Throughout this paper, Z^+ will denote the set of all positive integers. Let us denote by $E(S)$ the set of all idempotents of a semigroup S. An element a of a semigroup S with zero 0 is *nilpotent* if there exists $n \in Z^+$ such that $a^n = 0$. A semigroup S is a *nil-semigroup* if all of elements of S are nilpotents. If $n \in Z^+$, then a semigroup S is *n-nilpotent* if $S^n = \{0\}$. An ideal extension S of a semigroup K is a *nil-extension* (*n-nilpotent extension*) of K if S/K is a nil-semigroup (n-nilpotent semigroup). A subsemigroup K of a semigroup S is a *retract* of S if there exists a homomorphism φ of S onto K such that $\varphi(a) = a$ for all $a \in K$. Such a homomorphism is called a *retraction.* An ideal extension S of K is a *retract extension* (or *retractive extension*) of K if K is a retract of S. A semigroup S is an *n-inflation* of a semigroup K if S is a $(n+1)$-nilpotent extension and a retractive extension of K.

For undefined notions and notations we refer to [1] and [5].

In the next considerations the following results will be used:

Proposition 1. [4] *Let τ and ρ be congruences on a semigroup S such that $\tau \subseteq \rho$. Then the relation ρ/τ defined on S/τ by*

$$(a\tau, b\tau) \in \rho/\tau \Leftrightarrow (a, b) \in \rho,$$

is a congruence and $(S/\tau)/(\rho/\tau) \cong S/\rho$.

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Proposition 2. [5] *Every retractive extension S of a semigroup K by a semigroup Q with zero is a subdirect product of K and Q.*

In this paper we will consider cases when the converse of Proposition 2 holds.

Lemma 1. *Let $S \subseteq K \times Q$ be a subdirect product of a semigroup K and a semigroup Q with zero 0 such that $K \times \{0\} \subseteq S$. Then S is isomorphic to a semigroup which is a retract extension of K by some semigroup Q' with zero. Moreover, Q is isomorphic to some factor semigroup of Q'.*

Proof. Let $S \subseteq K \times Q$ be a subdirect product of a semigroup K and a semigroup Q with zero 0 such that $K \times \{0\} \subseteq S$. Then $K \times \{0\} \cong K$ is an ideal of S. Let $\pi_1 : S \to K$ and $\pi : K \times \{0\} \to K$ be projections, τ the Rees congruence on S induced by the ideal $K \times \{0\}$ and ρ the congruence on S induced by the projection $\pi_2 : S \to Q$. Then it is clear that π is an isomorphism and π_1 is an epimorphism. Hence the mapping

$$\varphi = \pi_1 \pi^{-1} : S \to K \times \{0\}$$

is a retraction. Therefore, S is isomorphic to a semigroup which is a retractive extension of a semigroup K by a semigroup $Q' \cong S/\tau$ with zero.

Moreover, since $S/\rho \cong Q$, by Proposition 1 we obtain that

$$(S/\tau)/(\rho/\tau) \cong S/\rho \cong Q,$$

whence it follows that Q is isomorphic to some factor semigroup of the semigroup $Q' \cong S/\tau$.

Lemma 2. *Let K be a semigroup such that for every element $a \in K$ there exists an idempotent $e \in E(K)$ which is a left or right identity for a. Then a semigroup S is isomorphic to a retractive nil-extension of K if and only if S is a subdirect product of K and a nil-semigroup.*

Proof. Let $S \subseteq K \times Q$ be a subdirect product of a semigroup K and a nil-semigroup Q with zero 0. Let $a \in K$. Since S is a subdirect product of K and Q, there exists $u \in Q$ such that $(a, u) \in S$. Also, by hypothesis it follows that there exists $e \in E(K)$ such that $ea = a$ or $ae = a$. Assume that $ea = a$ (in a similar way we consider the case with $ae = a$). Then there exists $v \in Q$ such that $(e, v) \in S$. Moreover, there exists $n \in \boldsymbol{Z}^+$ such that $v^n = 0$. Now we obtain that

$$(e, 0) = (e^n, v^n) = (e, v)^n \in S$$

and

$$(a, 0) = (ea, 0u) = (e, 0)(a, u) \in S^2 \subseteq S.$$

Therefore, $K \times \{0\} \subseteq S$. By Lemma 1, it follows that S is isomorphic to a retractive extension of a semigroup K by some semigroup F with zero. Let $(a, u) \in S$. Then there exists $n \in \boldsymbol{Z}^+$ such that $(a, u)^n = (a^n, u^n) = (a^n, 0) \in K \times \{0\}$. Hence, F is a nil-semigroup.

The converse follows from Proposition 2.

Theorem 1. *A semigroup S is a retractive nil-extension of a regular semigroup K if and only if S is a subdirect product of K and a nil-semigroup.*

Proof. Since for every element of a regular semigroup there exists a pair of idempotents which are its left and right identities, respectively, this proof follows from Lemma 2.

Corollary 1. [6] *A semigroup S is a retractive nil-extension of a rectangular group if and only if S is a subdirect product of a group, a left zero semigroup, a right zero semigroup and a nil-semigroup.*

Corollary 2. *A semigroup S is an n-inflation of a regular semigroup K if and only if S is a subdirect product of K and a* $(n+1)$*-nilpotent semigroup.*

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