43. q-analogue of de Rham Cohomology Associated with Jackson Integrals. I

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In this note we want to give a new formulation of Jackson integrals involved in basic hypergeometric functions through the classical Barnes' representations. We define a q-analogue of de Rham cohomology which can be formulated by means of q-version of Sato's b-functions and derive associated holonomic q-difference system. The evaluation of its multiplicity will be given as a number of different asymptotics.

1. *Structure of b-functions.* We take the elliptic modulus $q=e^{2\pi i\tau}$, $\mathrm{Im}\,\tau>0$. Let X be an n dimensional integer lattice $\simeq \boldsymbol{Z}^n$. We put $\bar{X}=X\otimes\boldsymbol{C}^*$, the n dimensional algebraic torus twisted by q. Let $\chi_1, \chi_2, \cdots, \chi_n$ be a basis of X such that an arbitrary $\chi\in X$ can be uniquely written by $\chi=\sum_{j=1}^n \nu_j\chi_j$, $\nu_j\in\boldsymbol{Z}$. We may identify $\bar{X}$ isomorphic to $X\otimes(\boldsymbol{C}/(2\pi i/\log q))$ with the direct product of n pieces of $\boldsymbol{C}^*$. The inclusion $X\subset\bar{X}$ can be obtained by identifying χ_j with the element $t=(1, \cdots, 1, q, 1, \cdots, 1)\in(\boldsymbol{C}^*)^n$. We denote by Q_j the shift operator $Q_jf(t)=f(\chi_j\cdot t)$ induced by the displacement $t\to\chi_j\cdot t$ for a function f on $\bar{X}$. We put $Q^\chi=Q_1^{\nu_1}\cdots Q_n^{\nu_n}$. We consider the q-difference equations

$$Q^\chi\Phi(t)=b_\chi(t)\Phi(t), \qquad \chi\in X \text{ and } t\in\bar{X}, \tag{1.1}$$

for a set of rational functions $\{b_\chi(t)\}_{\chi\in X}$, on $\bar{X}$, which are not identically zero. $\{b_\chi(t)\}_{\chi\in X}$ satisfies the compatibility condition

$$b_{\chi+\chi'}(t)=b_\chi(t)\cdot Q^\chi b_{\chi'}(t), \tag{1.2}$$

so that $\{b_\chi(t)\}_{\chi\in X}$ defines a 1-cocycle on X with values in $R^\times(\bar{X})$ the multiplicative abelian group consisting of non-zero rational functions on $\bar{X}$. We denote by $R(\bar{X})$ the field of rational functions on $\bar{X}$. $\{b_\chi(t)\}_{\chi\in X}$ is a coboundary if and only if $b_\chi(t)=Q^\chi\varphi(t)/\varphi(t)$ for $\varphi\in R^\times(\bar{X})$. We write the corresponding 1-cohomology by $H^1(X, R^\times(\bar{X}))$.

We put $(x)_\infty=\prod_{\nu=0}^\infty(1-xq^\nu)$ and $(x)_n=(x)_\infty/(xq^n)_\infty$ for $n\in\boldsymbol{Z}$. Then the following important result holds.

Proposition. *An arbitrary cocycle* $\{b_\chi(t)\}_{\chi\in X}$ *modulo a coboundary can be expressed by* (1.1), *where* Φ *denotes a q-multiplicative function on* $\bar{X}$ *written by*

$$\Phi=\prod_{j=1}^n t_j^{\alpha_j}\prod_{j=1}^m\frac{(a'_jt^{\mu_j})_\infty}{(a_jt^{\mu_j})_\infty} \tag{1.3}$$

for some non-negative integer m and α_j, a'_j, $a_j\in\boldsymbol{C}$, *and for* $\mu_j\in\check{X}=\mathrm{Hom}(X, \boldsymbol{Z})$. t^{μ_j} *denotes a monomial* $t_1^{\mu_j(\chi_1)}\cdots t_n^{\mu_j(\chi_n)}$. a_j *or* a'_j *may vanish or may not.*

This is a q-version of Sato's theorem in [6] and can be proved in a completely similar way (see the appendix in [6]).

We shall assume from now on that *any of a_j and a'_j don't vanish.* If we replace μ_j, $a_j=q^{s_j}$ and $a'_j=q^{s'_j}$ by $-\mu_j$, qa'^{-1}_j and qa_j^{-1} respectively in the factors of Φ, then

(1.4) $$T_j\Phi=t^{(s_j-s'_j)\mu_j}\frac{(qa_j^{-1}t^{-\mu_j})_\infty(a_jt^{\mu_j})_\infty}{(a'_jt^{\mu_j})_\infty(q^{-1}a'^{-1}_jt^{-\mu_j})_\infty}\Phi$$

also satisfies the same equation (1.1) and may replace Φ if necessary.

It is convenient to write $\mu_{-j}=-\mu_j$, $a'_{-j}=qa_j^{-1}$, $a_{-j}=qa_j^{-1}$ for $j\in\{\pm 1, \cdots, \pm m\}$. We also put $u_j=q^{\alpha_j}$.

We denote by $\mathcal{L}=\boldsymbol{C}[t_1, t_1^{-1}, \cdots, t_n, t_n^{-1}]$ the Laurent polynomial ring in t. The function $b_\chi(t)$ can be expressed by $u^\chi(b_\chi^+(t)/b_\chi^-(t))$ for $u^\chi=u_1^{\nu_1}\cdots u_n^{\nu_n}$ and $b_\chi^\pm(t)\in\mathcal{L}$, where $b_\chi^\pm(t)$ denote

(1.5) $$b_\chi^+(t)=\prod_{\mu_j(\chi)>0}(a_jt^{\mu_j})_{\mu_j(\chi)}\cdot\prod_{\mu_j(\chi)<0}(a'_jq^{\mu_j(\chi)}t^{\mu_j})_{-\mu_j(\chi)},$$

(1.6) $$b_\chi^-(t)=\prod_{\mu_j(\chi)>0}(a'_jt^{\mu_j})_{\mu_j(\chi)}\cdot\prod_{\mu_j(\chi)<0}(a_jq^{\mu_j(\chi)}t^{\mu_j})_{-\mu_j(\chi)}$$

respectively.

2. *Jackson integral and q-analogue of de Rham cohomology.* We denote by $\varpi=d_qt_1/t_1\wedge\cdots\wedge d_qt_n/t_n$ the canonical invariant n-form on $\bar{X}$. We consider the Jackson integral $\tilde{f}=\int_{X\cdot\xi}f\varpi$ for a function f on $\bar{X}$ over an orbit $X\cdot\xi$, $\xi\in\bar{X}$ as follows:

(2.1) $$\tilde{f}=(1-q)^n\sum_{\chi\in X}Q^\chi f(\xi),$$

if it is summable. We denote by $\langle\varphi\rangle$ the Jackson integral $\widetilde{\Phi\varphi}$. Then by definition, we have the equality $\tilde{f}=\widetilde{Q^\chi f}$ which is independent of the choice of the point ξ. If $f=\Phi\varphi$, $\varphi\in R(\bar{X})$, then

(2.2) $$\langle\varphi-b_\chi\cdot Q^\chi\varphi\rangle=0,\qquad \chi\in X.$$

In particular,

(2.3) $$\langle\varphi-b_{\chi_j}\cdot Q_j\varphi\rangle=0,\qquad 1\le j\le n.$$

Definition 1. The operators $\nabla_j=1-b_{\chi_j}Q_j$, $1\le j\le n$, define a covariant q-differenciation ∇ on $\bar{X}$. They commute each other:

(2.4) $$\nabla_j\nabla_k=\nabla_k\nabla_j,\qquad 1\le j,\ k\le n,$$

because of the compatibility condition for $\{b_\chi(t)\}_{\chi\in X}$. It should be noted that this gives a q-analogue version of ordinary integrable covariant differentiations investigated in [4] (see also [1]).

Definition 2. We denote by $Q_{u_j}^{\pm1}=\tilde{Q}_j^{\pm1}$, $Q_{a_j}^{\pm1}$ and $Q_{a'_j}^{\pm1}$ the operators for a function of u_j, a_j and a'_j induced by the displacements $u_j\to u_jq^{\pm1}$, $a_j\to a_jq^{\pm1}$ and $a'_j\to a'_jq^{\pm1}$ respectively, i.e. $Q_{u_j}^{\pm1}\Phi=t_j^{\pm1}\Phi$, $1\le j\le n$; $Q_{a_j}\Phi=(1-a_jt^{\mu_j})\Phi$, $Q_{a_j}^{-1}\Phi=(1-a_jq^{-1}t^{\mu_j})^{-1}\Phi$, $Q_{a'_j}\Phi=(1-a'_jt^{\mu_j})^{-1}\Phi$, $Q_{a'_j}^{-1}\Phi=(1-a'_jq^{-1}t^{\mu_j})\Phi$, $1\le j\le m$ respectively.

Let $\mathcal{A}$ be the commutative algebra over $\boldsymbol{C}$ of operators generated by $Q_{u_j}^{\pm1}$, $Q_{a_j}^{\pm1}$ and $Q_{a'_j}^{\pm1}$. We define the subspace V of $R(\bar{X})$ as follows:

(2.5) $$V=\{A\Phi/\Phi\,|\,A\in\mathcal{A}\}.$$

Then the space $\Phi\cdot V$ is left invariant under $\mathcal{A}$. Moreover V is invariant under the covariant q-differenciation ∇^χ, $\chi\in X$. V contains $\mathcal{L}$. It is actually spanned by the rational functions φ

(2.6) $$\varphi=\frac{\bar{\varphi}}{\prod_{j=1}^{m}(a'_j t^{\mu_j})_{l'_j}\cdot\prod_{j=1}^{m}(a_j q^{-l_j}t^{\mu_j})_{l_j}},\qquad \bar{\varphi}\in\mathcal{L},$$

for $l_j\geq 0$ and $l'_j\geq 0$. The space $\Phi\cdot V$ is left invariant under the covariant q-differenciation ∇^{χ}. This suggests us to define the following Koszul complex:

Definition 3. (q-analogue of de Rham complex). We put $\Omega^{\cdot}=\sum_{r=0}^{n}\Omega^r$, for $\Omega^r=\wedge^r\check{X}\otimes V$. Let $e_1,\cdots,e_n$ be a basis of $\check{X}$ and $e_{i_1}\wedge\cdots\wedge e_{i_r}$ be a basis of $\wedge^r\check{X}$. An arbitrary element of Ω^r can be represented by $\{\varphi_{i_1\cdots i_r}\}_{i_1<\cdots<i_r}$ through $(e_{i_1}\wedge\cdots\wedge e_{i_r})\otimes\varphi_{i_1\cdots i_r}\in\Omega^r$, $\varphi_{i_1\cdots i_r}\in V$. The boundary operation from Ω^r into Ω^{r+1} is given by

(2.7) $$(\nabla\varphi)_{i_1,\cdots,i_{r+1}}=\sum_{\nu=1}^{r+1}(-1)^{\nu-1}\nabla_{i_\nu}\varphi_{i_1,\cdots,i_{\nu-1},i_{\nu+1},\cdots,i_{r+1}}.$$

Then we have $\nabla^2=0$, because of (2.4). Hence we can define its cohomology $H^*(\Omega^{\cdot},\nabla)=\sum_{r=0}^{n}H^r(\Omega^{\cdot},\nabla)$. In particular we have the n-th cohomology $H^n(\Omega^{\cdot},\nabla)$ which is isomorphic to

(2.8) $$V/\sum_{j=1}^{n}(1-b_{\chi_j}Q_j)V=V/\sum_{\chi\in X}(1-b_\chi Q^\chi)V.$$

It is important to note that $\langle\varphi\rangle$ vanishes for $\varphi\in\nabla\Omega^{n-1}$ by (2.2).

Under these circumstances we may pose the following questions:

Q 1. *Is* $\dim H^*(\Omega^{\cdot},\nabla)<\infty$?

Q 2. *What is the dual of* $H^*(\Omega^{\cdot},\nabla)$? *Is it constructed in a geometric way as a family of countable sets in* $\bar{X}$? *If they exist, we may call them q-cycles.*

Q 3. *Do* $H^r(\Omega^{\cdot},\nabla)$ *vanish for all* $r<n$?

Q 4. *What is the Euler number* $\sum_{r=0}^{n}(-1)^r\dim H^r(\Omega^{\cdot},\nabla)$?

Under suitable assumptions one may conjecture that it is equal to $(-1)^n\kappa$, *for*

(2.9) $$\kappa=\sum_{i_1<i_2<\cdots<i_n}[\mu_{i_1},\cdots,\mu_{i_n}]^2,$$

where $[\mu_{i_1},\cdots,\mu_{i_n}]$ *denotes the determinant* $\det(\mu_{i_r}(\chi_s))_{1\leq r,s\leq n}$ *of the* $i_1,\cdots,i_n$ *th elements among* $\mu_1,\cdots,\mu_m$.

If Q 3 and Q 4 are affirmative, then $\dim H^n(\Omega^{\cdot},\nabla)=\kappa$.

3. *Holonomic q-difference equations.* We fix a generic $\eta\in X$. $\tilde{\Phi}$ is quasi-meromorphic in $u\in(\mathbf{C}^*)^n$ and satisfies the system of linear q-difference equations $(\mathcal{E})$:

(3.1) $$(b_\chi^-(\tilde{Q})u^{-\chi}-b_\chi^+(\tilde{Q}))\tilde{\Phi}=0,\qquad \text{for } \chi\in X.$$

This is equivalent to the subsystem $(\mathcal{E}^+)$:

(3.2) $$(b_\chi^-(\tilde{Q})u^{-\chi}-b_\chi^+(\tilde{Q}))\tilde{\Phi}=0,\qquad \text{for } \chi\in X,$$

such that $(\eta,\chi)>0$.

If $\tilde{\Phi}$ has an asymptotic behaviour

(3.3) $$\tilde{\Phi}\sim u_1^{\lambda_1}\cdots u_n^{\lambda_n}\left(1+O\left(\frac{1}{N}\right)\right),$$

for $\alpha=\eta N+\alpha'$, $N\to+\infty$, then q^λ must satisfy

(3.4) $$b_\chi^-(q^{\lambda-\chi})=0,\qquad \text{for each } \chi \text{ such that } (\eta,\chi)>0.$$

Assume that all the zeros of (3.4) *are isolated in* $\bar{X}$ *and X-inequivalent to each other.* Then their number equals κ and there exist κ asymptotic solutions of $(\mathcal{E})$. These are given by the Jackson integrals $\tilde{\Phi}$ over certain q-cycles containing each q^{λ}.

The system $(\mathcal{E})$ consists of an infinite number of equations which contain redundant ones of the form (3.1). We can reduce them by using the following

Lemma. *Fix χ' and $\chi'' \in X$. Assume $\mu_j(\chi')\mu_j(\chi'') \geq 0$ for all j. Then an arbitrary quasi-meromorphic function f of $u \in \check{X} \otimes C^*$ satisfying* (3.1) *with $\chi = \chi'$ and χ'' satisfies* (3.1) *too with $\chi = \chi' + \chi''$.*

Indeed then $b^{\pm}_{\chi+\chi'}(t) = b^{\pm}_{\chi}(t) \cdot Q^{\chi} b^{\pm}_{\chi'}(t)$.

Our problem is intimately related to the torus embeddings (see [5]). Let F be a fan divided by hyperplanes H_j: $\mu_j(\omega) = 0$, $\omega \in X_R$ for $X_R = X \otimes \boldsymbol{R}$. F consists of rational polyhedral cones σ given by the connected components of the complement $X_R - \bigcup_{j=1}^{m} H_j$. It is known that F corresponds to a torus embedding $T_{emb}(F)$ which is a compactification of the algebraic torus $\bar{X}$. There exists a fan F^* which is a simplicial subdivision of F such that each cone composing F^* is generated by a basis of X. It is known that the torus embedding $T_{emb}(F^*)$ gives a desingularization of $T_{emb}(F)$ and vice versa. We denote by Y the set of corner elements generating rational polyhedral cones in F^*. Then $(\mathcal{E}^+)$ *are equivalent to the system of a finite number of q-difference equations* $(\mathcal{E}^+_Y)$:

$$(b^-_{\chi}(\tilde{Q})u^{-\chi} - b^+_{\chi}(\tilde{Q}))\tilde{\Phi} = 0, \tag{3.5}$$

for $\chi \in Y$ such that $(\eta, \chi) > 0$. Then we have

Theorem. *$\tilde{\Phi}$ satisfies the system of q-difference equations $(\mathcal{E}^+_Y)$. $(\mathcal{E}^+_Y)$ has κ linearly independent solutions which have asymptotic behaviours* (3.3) *satisfying* (3.4) *in a generic direction $\eta \in \check{X}$. These solutions are given by the Jackson integrals over κ q-cycles containing q^{λ} satisfying* (3.4).

In the second part, under more restrictive conditions, we shall construct such κ q-cycles and show that $\dim H^n(\Omega^{\cdot}, \nabla)$ equals κ, by using the notions of *q-analogue of stable cycles, Newton polyhedra and torus embeddings.*

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