40. A Note on the Mean Value of the Zeta and L-functions. VII

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(Communicated by Kunihiko KODAIRA, M. J. A., June 12, 1990)

1. Let $E_2(T)$ be the error-term in the asymptotic formula for the fourth power mean of the Riemann zeta-function, so that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = TP_4(\log T)+E_2(T) \tag{1}$$

with a certain polynomial P_4 of degree 4. As has been pointed out already in the preceding note [3] of this series, Corollary 2 in it gives an alternative proof of Zavorotnyi's claim [5]:

$$E_2(T)=O(T^{(2/3)+\varepsilon}) \tag{2}$$

for any fixed $\varepsilon>0$. In fact this is simply a resultant of combining the corollary with the spectral mean of Hecke series ([4]):

$$\sum_{\kappa_j\leq x} \alpha_j H_j\left(\frac{1}{2}\right)^4 \ll x^{2+\varepsilon}. \tag{3}$$

Here $\{\lambda_j=\kappa_j^2+(1/4), \kappa_j>0\}$ is the discrete spectrum of the non-Euclidean Laplacian on $SL(2, Z)$, and $\alpha_j=|\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1}$ with the first Fourier coefficient $\rho_j(1)$ of the Maass wave form corresponding to λ_j to which the Hecke series H_j is attached.

Though (2) is the hitherto best result on the upper bound for $E_2(T)$ it is generally believed that $T^{(1/2)+\varepsilon}$ may be the true order of it. The aim of the present note is to study this problem from the opposite direction. Namely we are going to show that under a hypothesis of the type of nonvanishing theorems for automorphic L-functions one may deduce an Ω-result on $E_2(T)$ from [Theorem, 3].

To formulate the hypothesis we denote by $\{\mu_h\}$, arranged in the increasing order, the mutually different members in the set $\{\kappa_j\}$. And we put

$$G_h=\sum_{\kappa_j=\mu_h} \alpha_j H_j\left(\frac{1}{2}\right)^3.$$

Now we set out

Hypothesis A. *Not all G_h vanish.*

Then we have

Theorem. *Under Hypothesis A $E_2(T)=\Omega(T^{1/2})$ holds.*

We note that Zavorotnyi's argument [5] does not seem to be able to yield our theorem. Also the theorem should be compared with the Ω-result on the mean square of $|\zeta((1/2)+it)|$ due to Good [1] (for the latest develop-

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ments see Hafner and Ivić [2]).

Remark. (i) If $G_h=0$ for all h then the Lindelöf hypothesis for $\zeta(s)$ would follow. (ii) It is clear that Hypothesis A is a consequence of the following stronger statement:

Hypothesis B. *There exists a* λ_j *of multiplicity one such that* $H_j(1/2)\neq 0$.

Probably this may be checked numerically.

2. We shall give an outline of our proof of the theorem; a detailed version is available in a form of manuscript, and will be published elsewhere.

As in [3] we put

$$I_4(T,\Delta)=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}dt,$$

and consider the expression

$$\int_T^{2T} I_4(t,\Delta)\,dt=TQ_4(\log T)+R(T,\Delta), \tag{4}$$

where Q_4 is a polynomial of degree 4. This is suggested by [Corollary 2, 3], so that $R(T,\Delta)$ is supposed to be of the order of T^c with a $c<1$ for some suitably chosen Δ. Then we make the initial observation:

Lemma 1. *If* $R(T,\Delta)=\Omega(g(T))$ *for* Δ *satisfying* $T^{\varepsilon}\leqq\Delta\leqq T^{1-\varepsilon}$ *and* $\Delta=o(g(T)\log^{-5}T)$ *then we have* $E_2(T)=\Omega(g(T))$.

This can be proved by truncating, in an obvious manner, the double integral involved in (4) and taking (1) into account.

Next we introduce

$$S(V,\Delta)=\int_V^{2V} R(T,\Delta)\,dT.$$

If Hypothesis A implies $S(V,\Delta)=\Omega(V^{3/2})$ for Δ satisfying $V^{\varepsilon}\leqq\Delta\leqq V^{(1/2)-\varepsilon}$ then obviously $R(T,\Delta)=\Omega(T^{1/2})$ for Δ satisfying $T^{\varepsilon}\leqq\Delta\leqq T^{(1/2)-\varepsilon}$, and Lemma 1 ends the proof of the theorem. Thus we are led to the problem of finding a suitable explicit formula for $S(V,\Delta)$ for Δ in the indicated range. The reason that we have integrated $R(T,\Delta)$, instead of treating it directly, is that for $S(V,\Delta)$ we can give such an explicit formula but it seems difficult to do so for $R(T,\Delta)$.

In fact, after a somewhat involved computation we have deduced from [Theorem, 3] and (3) the following:

Lemma 2. *Uniformly for* $V^{\varepsilon}\leqq\Delta\leqq V^{(1/4)-\varepsilon}$ *we have*

$$S(V,\Delta)=2V^{3/2}\,\mathrm{Im}\left\{\sum_{h=1}^{\infty}G_h(F(\mu_h)V^{i\mu_h}+F(-\mu_h)V^{-i\mu_h})\right\}+O(V^{3/2}(\log V)^{-1}),$$

where

$$F(\mu)=\exp\left(\frac{\pi}{2}\left(\mu+\frac{i}{2}\right)\right)((\sinh\pi\mu)^{-1}+i)\frac{\Gamma^3((1/2)-i\mu)(2^{(1/2)+i\mu}-1)(2^{(3/2)+i\mu}-1)}{\Gamma(1-2i\mu)(1+2i\mu)(3+2i\mu)}.$$

We note that $F(\mu)=O(|\mu|^{-5/2})$ as $|\mu|$ tends to infinity; this and (3) imply that the last sum over h is absolutely convergent.

Now, providing Hypothesis A, the desired Ω-result for $S(V,\Delta)$ will be an easy consequence of the last lemma, once we establish the following general assertion:

Lemma 3. *Let* $\{a_j\}$ *and* $\{b_j\}$ *be such that* $|a_1|>|b_1|$ *and* $\sum_{j=1}^{\infty}(|a_j|+|b_j|)<\infty$. *Also let* $\{\omega_j\}$ *be a strictly increasing sequence of positive numbers. Then we have, as* x *tends to infinity,*

$$\mathrm{Im}\left\{\sum_{j=1}^{\infty}(a_j x^{i\omega_j}+b_j x^{-i\omega_j})\right\}=\Omega_{\pm}(1).$$

To show this we denote the left side by $\varphi_0(x)$ and define $\varphi_n(x)$ inductively by

$$\varphi_{n+1}(x)=\int_x^{\tau x}\varphi_n(x)\frac{dx}{x},$$

where $\tau=\exp(\pi/\omega_1)$. As is easily seen we have, for any $n\geqq 0$,

$$\left|2^{-n}\varphi_n(x)-\mathrm{Im}\left\{a_1\left(\frac{i}{\omega_1}\right)^n x^{i\omega_1}\right\}\right|\leqq|b_1|\omega_1^{-n}+\sum_{j=2}^{\infty}(|a_j|+|b_j|)\omega_j^{-n}.$$

Taking n sufficiently large the right side is less than $|a_1|\omega_1^{-n}$, but the member in the braces can be equal to both $i|a_1|\omega_1^{-n}$ and $-i|a_1|\omega_1^{-n}$ infinitely often. This proves the lemma and thus the theorem.

Remark. Lemma 2 has an obvious counterpart in the theory of prime numbers. That is, in just the same way as the Chebyshev function $\psi(x)$ is related to all complex zeros of $\zeta(s)$ the fourth power moment of $|\zeta((1/2)+it)|$ is related to all eigenvalues of the non-Euclidean Laplacian on $SL(2,Z)$ (or more exactly, all complex zeros of Selberg's zeta-function for $SL(2,Z)$).

Addendum. After submitting this paper we learned that A. Good (J. Number Theory, **13**, 18–65 (1981)) had obtained an Ω-result of the same strength as ours for Hecke L-series associated with holomorphic cusp forms under a certain hypothesis of the type of non-vanishing theorems.

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