75. A Note on the Mean Value of the Zeta and L-functions. VI

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Continuing our previous work [3] we show an explicit formula for

$$I_4(T, \Delta)=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}dt.$$

We retain the notations introduced in [3] from the theory of automorphic functions.

We define $D_\Delta(r, s)$ as the analytic continuation of

$$\int_0^\infty x^{r-1}(1+x)^{-s}\exp\left(-\left(\frac{\Delta}{2}\log(1+x)\right)^2\right)dx,$$

and put

$$\Psi(u, v, w, z; \xi)=-i(2\pi)^{z-v-2}\cos\left(\frac{\pi}{2}(u-w)\right)$$
$$\times\int_{-\infty i}^{\infty i}\sin\left(\frac{\pi}{2}(u+v+w+z-2r)\right)\Gamma\left(\frac{1}{2}(u+v+w+z-1)+\xi-r\right)$$
$$\times\Gamma\left(\frac{1}{2}(u+v+w+z-1)-\xi-r\right)\Gamma(1-w-z+r)\Gamma(1-u-z+r)D_\Delta(r, z)dr.$$

The path of integration is curved to ensure that the poles of the first three factors on the integrand lie on the right of the path and those of the remaining factors on the left; it is assumed that u, v, w, z, ξ are such that the contour can be drawn. Also we define $\Phi(u, v, w, z; \xi)$ to be the one which is obtained by replacing in the above the factors $\cos((\pi/2)(u-w))$ and $\sin((\pi/2)(u+v+w+z-2r))$ by $\cos(\pi\xi)$ and $\cos((\pi/2)(u+w+2z-2r))$, respectively. It can be shown that Ψ and Φ admit meromorphic continuations over the entire C^5; hereafter the symbols Ψ and Φ will denote these meromorphic functions. Ψ and Φ are of rapid decay: Uniformly for any bounded u, v, w, z and for any fixed $c>0$ we have $\Psi=O(|\xi|^{-c}e^{-\pi|\xi|})$ and $\Phi=O(|\xi|^{-c})$ when $|\mathrm{Im}\,\xi|$ tends to infinity while $\mathrm{Re}\,\xi$ remains bounded. Further we put, with an obvious abuse of notation,

$$\Theta(\xi; T, \Delta)=2\,\mathrm{Re}\,\{(\Psi-\Phi)(P_T; i\xi)\},$$
$$\Lambda(k; T, \Delta)=2\,\mathrm{Re}\left\{\Psi\left(P_T; k-\frac{1}{2}\right)\right\},$$

where P_T is the point $(\frac{1}{2}+iT, \frac{1}{2}-iT, \frac{1}{2}+iT, \frac{1}{2}-iT)$, $T>0$, and $k=1, 2, 3\cdots$. We should note that for any fixed $B>0$ $\Lambda(k; T, \Delta)=O(\Delta^{-k})$ if $k\leq B$, and $=O((k\Delta)^{-B})$ if $k>B$, where the implied constants depend only on B.

Then our main result is as follows:

Theorem. *If $0<\Delta<T(\log T)^{-1}$ we have*

$$I_4(T,\Delta)=F_0(T,\Delta)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+i\xi)|^6}{|\zeta(1+2i\xi)|^2}\Theta(\xi\,;T,\Delta)d\xi$$

$$(1)\qquad +\sum_{j=1}^{\infty}\alpha_jH_j\left(\frac{1}{2}\right)^3\Theta(\kappa_j\,;T,\Delta)+\sum_{k=6}^{\infty}\sum_{j=1}^{d_{2k}}\alpha_{j,2k}H_{j,2k}\left(\frac{1}{2}\right)^3\Lambda(k\,;T,\Delta)+O(T^{-1}\log T),$$

where, with certain absolute constants $c(a,b\,;f,g)$, $a,b,f,g\geqq 0$,

$$F_0(T,\Delta)=\mathrm{Re}\left\{(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\sum_{af+bg\leqq 4}c(a,b\,;f,g)\left(\frac{\Gamma^{(a)}}{\Gamma}\right)^f\left(\frac{\Gamma^{(b)}}{\Gamma}\right)^g\left(\frac{1}{2}+i(T+t)\right)\times e^{-(t/\Delta)^2}dt\right\},$$

and the implied constant in the error-term is absolute.

The above should be compared with the following explicit formula for the mean square of $\zeta(s)$: For any Δ, $T>0$,

$$(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^2e^{-(t/\Delta)^2}dt$$

$$(2)\qquad \begin{aligned}&=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+i(T+t)\right)e^{-(t/\Delta)^2}dt+2\gamma-\log(2\pi)-2\frac{\sqrt{\pi}}{\Delta}\\&\times\exp\left(\frac{1}{\Delta^2}\left(\frac{1}{4}-T^2\right)\right)\cos\left(\frac{T}{\Delta^2}\right)+4\sum_{n=1}^{\infty}d(n)\int_0^{\infty}\frac{\sin(2\pi nx)}{\sqrt{x(1+x)}}\\&\times\cos\left(T\log\left(1+\frac{1}{x}\right)\right)\exp\left(-\left(\frac{\Delta}{2}\log\left(1+\frac{1}{x}\right)\right)^2\right)dx.\end{aligned}$$

From this we can deduce Atkinson's formula [1, Theorem]. In fact, as Atkinson did in a bit different context, we apply Voronoi's formula to the last sum, getting a uniform convergence with respect to $\Delta\geqq 0$. Then taking the limit of both sides as $\Delta\to 0$ and integrating the result we obtain Atkinson's formula.

Now, comparing (1) with (2) the outward similarity may give rise to the problem of finding a sum formula of Voronoi type for the factors $\alpha_jH_j(\frac{1}{2})^3$ and $\alpha_{j,2k}H_{j,2k}(\frac{1}{2})^3$. If the analogy between (1) and (2) holds in the most optimistic way, then we would be able to deduce from (1) a complete ζ^4-version of Atkinson's formula.

Though we are unable to solve this we can deduce from (1) two consequences on the asymptotic behaviour of the fourth power mean of $\zeta(s)$ which enhance the analogy between (1) and (2):

Corollary 1. *If $T^{1/2}\leqq\Delta\leqq T(\log T)^{-1}$ then we have*

$$I_4(T,\Delta)=\frac{\pi}{\sqrt{2T}}\sum_{j=1}^{\infty}\alpha_j\kappa_j^{-1/2}H_j\left(\frac{1}{2}\right)^3\sin\left(\kappa_j\log\frac{\kappa_j}{4eT}\right)e^{-(\Delta\kappa_j/2T)^2}+O(T^{\varepsilon}).$$

Corollary 2. *Let*

$$J_4(V,\Delta)=\int_0^V I_4(T,\Delta)dT.$$

Then we have, for $V^{1/2}\leqq\Delta\leqq V(\log V)^{-1}$,

$$J_4(V,\Delta)=VP_4(\log V)+\pi\left(\frac{V}{2}\right)^{1/2}\sum_{j=1}^{\infty}\alpha_j\kappa_j^{-3/2}H_j\left(\frac{1}{2}\right)^3\cos\left(\kappa_j\log\frac{\kappa_j}{4eV}\right)e^{-(\Delta\kappa_j/2V)^2}+O(V^{(1/2)+\varepsilon}),$$

where P_4 *is a certain polynomial of degree* 4.

Corollary 1 implies Iwaniec's result [2, Theorem 4] and Corollary 2 yields an alternative proof of Zavorotnyi's claim [5].

A detailed proof of our result is available in [4].

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