83. A Note on the Artin Map

By Takashi ONO
Department of Mathematics, The Johns Hopkins University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1989)

Let K/k be a finite Galois extension of algebraic number field with the Galois group $G=G(K/k)$, $\mathfrak{p}$ a prime ideal of k unramified for K/k and $\mathfrak{P}$ be a prime factor of $\mathfrak{p}$ in K. Denote by $\left[\frac{K/k}{\mathfrak{P}}\right]$ the Frobenius automorphism of $\mathfrak{P}$. For an element $\sigma \in G$, denote by $C(\sigma)$ the conjugate class containing σ, by $h(\sigma)$ the cardinality of $C(\sigma)$ and by $a(\sigma)$ the following element in the center $\boldsymbol{C}[G]_0$ of the group ring $\boldsymbol{C}[G]$:

$$a(\sigma)=\frac{1}{h(\sigma)}\sum_{\tau\in C(\sigma)}\tau. \tag{1}$$

For $\sigma=\left[\frac{K/k}{\mathfrak{P}}\right]$, we may write, without ambiguity, $C_{\mathfrak{p}}$, $h_{\mathfrak{p}}$, $a_{\mathfrak{p}}$, instead of $C(\sigma)$, $h(\sigma)$, $a(\sigma)$, respectively. One verifies easily that

$$a_{\mathfrak{p}}=\frac{1}{g_{\mathfrak{p}}}\sum_{\mathfrak{P}\mid\mathfrak{p}}\left[\frac{K/k}{\mathfrak{P}}\right]=\frac{1}{n}\sum_{\sigma\in G}\left[\frac{K/k}{\mathfrak{P}^{\sigma}}\right], \qquad n=[K:k], \tag{2}$$

where $g_{\mathfrak{p}}$ means the number of distinct prime factors of $\mathfrak{p}$ in K. We shall denote by $\alpha_{K/k}(\mathfrak{p})$ the element in $\boldsymbol{C}[G]_0$ defined by any member of the equalities (2). When K/k is abelian, $\alpha_{K/k}(\mathfrak{p})$ is an element of G and we have

$$\alpha_{K/k}(\mathfrak{p})=\left(\frac{K/k}{\mathfrak{p}}\right) \qquad \text{(Artin symbol)}. \tag{3}$$

Back to any Galois extension K/k, put

$$I(K/k)=\{\mathfrak{a}\,;\ \text{ideal } (\neq 0) \text{ in } \mathfrak{o}_k,\ (\mathfrak{a}, \Delta_{K/k})=1\}, \tag{4}$$

where $\mathfrak{o}_k$ is the ring of integers of k and $\Delta_{K/k}$ denotes the relative discriminant of K/k. If

$$\mathfrak{a}=\prod_{\mathfrak{p}}\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{a})}, \qquad \mathfrak{a}\in I(K/k), \tag{5}$$

is the factorization of $\mathfrak{a}$ in k, we put

$$\alpha_{K/k}(\mathfrak{a})=\prod_{\mathfrak{p}}\alpha_{K/k}(\mathfrak{p})^{\nu_{\mathfrak{p}}(\mathfrak{a})}. \tag{6}$$

The map $\alpha_{K/k}$ whose domain of definition is now $I(K/k)$ is, as is easily seen, a homomorphism of the multiplicative semigroup $I(K/k)$ into the multiplicative semigroup of the commutative ring $\boldsymbol{C}[G]_0$ sending the identity $\mathfrak{o}_k$ to the identity 1_G. When K/k is abelian, the image of $\alpha_{K/k}$ is just the group G (by the density theorem due to Tschebotareff) and the determination of fibres of $\alpha_{K/k}$ is the content of the Artin reciprocity in class field theory. Therefore it is natural to study the image and fibres of the map $\alpha_{K/k}: I(K/k)\to\boldsymbol{C}[G]_0$ for nonabelian Galois extension K/k. Since the cardinality of the image of $\alpha_{K/k}$ is the order of G when K/k is abelian, let us start our study of $\alpha_{K/k}$ with a criterion for the finiteness of the image. To do this, we need some notations in character theory.

Since $C[G]$ is semisimple, there is an isomorphism

(7) $$C[G]\approx C_{n_1}\oplus\cdots\oplus C_{n_r},$$

where C_m denotes the ring of all matrices of order m over C. The isomorphism (7) induces an isomorphism

(8) $$\omega:\ C[G]_0 \xrightarrow{\sim} C^r.$$

Let ω_ν be the projection of ω on the νth factor and χ_ν be the irreducible characters of $C[G]$, $1\leqq\nu\leqq r$. Then, we have

(9) $$\chi_\nu(z)=n_\nu\omega_\nu(z), \qquad n_\nu=\chi_\nu(1),\ z\in C[G]_0.$$

From (1), (9), it follows that

(10) $$\omega_\nu(a(\sigma))=\frac{1}{n_\nu}\chi_\nu(\sigma), \qquad \sigma\in G,\ 1\leqq\nu\leqq r,$$

and

(11) $$|\omega_\nu(a(\sigma))|\leqq 1, \qquad \sigma\in G,\ 1\leqq\nu\leqq r.$$

Let σ_i, $1\leqq i\leqq r$, $\sigma_1=1$, be the representatives of conjugate classes of G. Hence (10) can be written

(12) $$\omega_\nu(a(\sigma_i))=\frac{1}{n_\nu}\chi_\nu(\sigma_i), \qquad 1\leqq\nu,\ i\leqq r.$$

Since the isomorphism ω in (8) induces homomorphisms ω_ν, $1\leqq\nu\leqq r$, we have, in view of (6),

(13) $$\omega_\nu(\alpha_{K/k}(\mathfrak{a}))=\prod_{\mathfrak{p}}\omega_\nu(\alpha_{K/k}(\mathfrak{p}))^{\nu_{\mathfrak{p}}(\mathfrak{a})}, \qquad 1\leqq\nu\leqq r.$$

Theorem. *Notations being as above, the image of the map $\alpha_{K/k}$ for a Galois extension K/k is finite if and only if $|\chi_\nu(\sigma_i)|=0$ or n_ν for all ν, i, $1\leqq\nu$, $i\leqq r$.*

Proof. 'if'-part. Assume the condition on characters. For a fixed ν, let $\varepsilon_1,\cdots,\varepsilon_{n_\nu}$ be the characteristic roots of the matrix $R_\nu(\sigma_i)$ where R_ν is a representation of G affording the irreducible character χ_ν. If $\chi_\nu(\sigma_i)\neq 0$, then we have $|\chi_\nu(\sigma_i)|=|\varepsilon_1+\cdots+\varepsilon_{n_\nu}|=n_\nu$ and so $\varepsilon_1=\cdots=\varepsilon_{n_\nu}=\varepsilon$, an nth root of 1, $n=[K:k]$. Hence $\omega_\nu(a(\sigma_i))=(1/n_\nu)\chi_\nu(\sigma_i)$ is an nth root of 1. Therefore all values $\omega_\nu(\alpha_{K/k}(\mathfrak{p}))$ and hence all values $\omega_\nu(\alpha_{K/k}(\mathfrak{a}))$ are either 0 or nth roots of 1. 'only if'-part. Suppose that $|\chi_\nu(\sigma_i)|\neq 0$, n_ν, for some ν, i. By the density theorem of Tschebotareff, there is a prime ideal $\mathfrak{P}$ in K which is unramified for K/k such that $\sigma_i=\left[\frac{K/k}{\mathfrak{P}}\right]$. Then, by (12), we have $0<|\omega_\nu(\alpha_{K/k}(\mathfrak{p}))|=(1/n_\nu)|\chi_\nu(\sigma_i)|<1$ and so, taking powers of $\mathfrak{p}$, we obtain infinitely many values of the map $\alpha_{K/k}$. Q.E.D.

Corollary. *Notations being as above, assume that all irreducible characters are Q-valued. Then, the image of $\alpha_{K/k}$ is finite if and only if n_ν divides $\chi_\nu(\sigma)$, this being an integer, for all ν, $1\leqq\nu\leqq r$, and all $\sigma\in G$.*

Remark 1. There exist two nonisomorphic nonabelian groups of order 8: D_4 (group of the symmetries of the square) and Q_8 (the quaternion group). They have the same character table $X=(\chi_\nu(\sigma_i))$. They have 4 linear characters χ_ν, $1\leqq\nu\leqq 4$, and exactly one other irreducible character χ_5 with $n_5=\chi_5(1)=2$. The character table is

$$X=\begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1 & 1\\ 1 & 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1 & 1\\ 2 & 0 & 0 & 0 & -2\end{pmatrix}. \tag{14}$$

Therefore, by the corollary above, the image of $\alpha_{K/k}$ is finite when $G=G(K/k)$ is nonabelian of order 8. The cardinality of the image of $\alpha_{K/k}$ is that of the image of $\omega\circ\alpha_{K/k}$ which is a set of vectors in $\boldsymbol{C}^r$. By (12), (13), (14), we see that those vectors are:

$$(1, (-1)^{e_2+e_4}, (-1)^{e_3+e_4}, (-1)^{e_2+e_3}, 0^{e_2+e_3+e_4}(-1)^{e_5})$$

with integers $e_i\geqq 0$, $2\leqq i\leqq 5$. Hence the image of $\alpha_{K/k}$ consists of 6 elements. If $G=S_3$ (symmetric group on 3 letters), one sees that the image $\alpha_{K/k}$ contains infinitely many elements. If we introduce, for any K/k, an equivalence relation in $I(K/k)$ by

$$\mathfrak{a} \underset{K/k}{\sim} \mathfrak{b} \overset{\text{def}}{\Longleftrightarrow} \alpha_{K/k}(\mathfrak{a})=\alpha_{K/k}(\mathfrak{b}), \tag{15}$$

and call $i(K/k)$ the cardinality of the quotient set $I(K/k)/\underset{K/k}{\sim}$, then $i(K/k)$ is, of course, equal to the cardinality of the image of $\alpha_{K/k}$. When K/k is abelian, $i(K/k)=[K:k]$ and so it may be interesting to look at the invariant $i(K/k)$ for nonabelian K/k, although $i(K/k)=\infty$ for many cases.

Remark 2. It is known (as Burnside theorem; see e.g. W. Feit, Characters of finite groups, Benjamin, 1967, p. 36, (6.9)) that if χ is a nonlinear irreducible character of a finite group G then $\chi(\sigma)=0$ for some $\sigma\in G$. Therefore, unlike the abelian case, one cannot extend the domain $I(K/k)$ of the map $\alpha_{K/k}$ to the group of fractional ideals prime to $\Delta_{K/k}$.

Remark 3. Denote by $\boldsymbol{C}[I(K/k)]$ the vector space (with infinite dimension) generated freely by all ideals in $I(K/k)$. As $\mathfrak{o}_k$ is a Dedekind ring, $\boldsymbol{C}[I(K/k)]$ is nothing but the ring of all polynomials over $\boldsymbol{C}$ with infinitely many variables $X_{\mathfrak{p}}$, $\mathfrak{p}\nmid\Delta_{K/k}$. By linearity, or by substituting $\alpha_{K/k}(\mathfrak{p})$ in $X_{\mathfrak{p}}$, we can extend $\alpha_{K/k}$ to a homomorphism of commutative $\boldsymbol{C}$-algebras:

$$\alpha_{K/k}:\ \boldsymbol{C}[I(K/k)]\longrightarrow \boldsymbol{C}[G]_0. \tag{16}$$

By the density theorem of Tschebotareff, we obtain the following short exact sequence:

$$0\longrightarrow \operatorname{Ker}\alpha_{K/k}\longrightarrow \boldsymbol{C}[I(K/k)]\longrightarrow \boldsymbol{C}[G]_0\longrightarrow 0. \tag{17}$$

The determination of the ideal $\operatorname{Ker}\alpha_{K/k}$ could be considered as an analogue of the Artin reciprocity.