43. *Estimates for Degenerate Schrödinger Operators and an Application for Infinitely Degenerate Hypoelliptic Operators*

By Yoshinori MORIMOTO
School of Mathematics, Yoshida College, Kyoto University

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1. Introduction and main theorems. In Chapter II of [1] Fefferman and Phong estimated the eigenvalues of Schrödinger operators $-\Delta+V(x)$ on R^n by using the uncertainty principle. Inspirated by their idea, in the present note we give two L^2-estimates for degenerate Schrödinger operators of higher order, which are a version and an extension of Theorem 4 in Chapter II of [1]. As an application, we consider the hypoellipticity for an example of infinitely degenerate elliptic operators.

Consider a symbol of the form

$$a(x,\xi)=\sum_{k=1}^{n} a_k(x)|\xi_k|^{2\mu_k}+V(x), \qquad x\in R^n, \tag{1}$$

where μ_k are positive rational numbers, $V(x)$ is a non-negative measurable function and

$$\begin{cases} a_1(x)=1, \\ a_k(x)=\prod_{j=1}^{k-1}|x_j|^{2\kappa(k,j)} \qquad \text{for } k\geq 2. \end{cases} \tag{2}$$

Here $\kappa(k,j)$ are non-negative rational numbers. If $(x_0,\xi_0)\in R^{2n}$ and if $\delta=(\delta_1,\cdots,\delta_n)$ for $\delta_j>0$, we denote by $B_\delta(x_0,\xi_0)$ a box

$$\{(x,\xi)\,;\,|x_j-x_{0j}|\leq\delta_j/2,\ |\xi_j-\xi_{0j}|\leq\delta_j^{-1}/2\}. \tag{3}$$

Clearly the volume of $B_\delta(x_0,\xi_0)$ is equal to 1. Let $\mathcal{C}$ denote a set of boxes $B_\delta(x_0,\xi_0)$ for all (x_0,ξ_0) and all δ. We denote by $\mathrm{m}_l(\cdot)$ the Lebesgue measure in R^l. We set $m_k=\mu_k-1$ if μ_k is integer and $m_k=[\mu_k]$ otherwise. Set $m_0=\sum_{k=1}^n m_k$.

Theorem 1. *Let $a(x,\xi)$ be the above symbol and let $W(x)$ be a continuous function in R^n. Assume that there exists a constant $1-2^{-m_0}<c\leq 1$ such that for any $B=B_\delta(x_0,\xi_0)\in\mathcal{C}$*

$$\mathrm{m}_{2n}(\{(x,\xi)\in B\,;\,a(x,\xi)\geq\max_{\pi(B^{**})}W(x)\})\geq c, \tag{4}$$

where π is a natural projection from $R^{2n}_{x,\xi}$ to R^n_x and B^{**} denotes a suitable dilation of B whose modulus depends only on μ_k and $\kappa(k,j)$. Then for any compact set K of R^n_x there exists a constant $c_K>0$ such that

$$(a(x,D)u,u)\geq c_K(W(x)u,u) \qquad \text{for any } u\in C_0^\infty(K), \tag{5}$$

where $(\ ,\ )$ denotes the L^2 inner product (cf. Theorem B in [5]).

Remark 1. The lower bound of c in (4) is 0 when all $\mu_k\leq 1$. If all $a_k(x)\equiv 1$ then the constant c_K in (5) can be taken independent of K. The theorem holds even if each variable x_j is replaced by the vector $\boldsymbol{x}_j=(x_1^j, \cdots, x_{l_j}^j)$. The rationality assumption of μ_k and $\kappa(k,j)$ can be removed.

In the polynomial potential case the theorem becomes fairly simple. In order to explain this fact, for a $0<h\leq 1$ we redefine a set $\mathcal{C}_h$ of boxes

(6) $$B_{\delta,h}(x_0,\xi_0)\equiv\{(x,\xi);|x_j-x_{0j}|\leq\delta_j/2,\ |\xi_j-\xi_{0j}|\leq h\delta_j^{-1}/2\}$$

for all (x_0,ξ_0) and all δ.

Theorem 2. *Let $a(x,\xi)$ be the symbol of the form* (1) *with $V(x)$ replaced by a polynomial $U(x)$ in R^n of order d, which is not always non-negative. Then for any compact set K of R^n there exists a positive $h=h_K\leq 1$ satisfying the following property: If the estimate*

(7) $$\max_{B_h} a(x,\xi)\geq 0$$

holds for any $B_h=B_{\delta,h}(x_0,\xi_0)\in\mathcal{C}_h$ then we have

(8) $$(a(x,D)u,u)\geq 0 \qquad \text{for any } u\in C_0^\infty(K).$$

Here the positive h depends only on d, n, μ_k and $\kappa(k,j)$ except K.

Remark 2. When all $a_k(x)\equiv 1$ then we can take $h>0$ independent of K. Furthermore, if all $\mu_k=1$ then Theorem 2 is nothing but one part of Theorem 4 in Chapter II of [1].

Remark 3. When $V(x)$ and $W(x)$ in Theorem 1 are polynomials, Theorem 1 follows from Theorem 2 by putting $U(x)=V(x)-h^{2\mu_0}W(x)$, where $\mu_0=\max_{1\leq k\leq n}\mu_k$. In fact, this is obvious if we note that for $0<h\leq 1$

$$\max_{B_h}\{a(x,\xi)-h^{2\mu_0}W(x)\}\geq h^{2\mu_0}\{\max_{B_1} a(x,\xi)-\max_{\pi(B_1)} W(x)\}.$$

2. Infinitely degenerate hypoelliptic operators. As an application of Theorem 1 we consider a second order elliptic operator with infinite degeneracy as follows:

(9) $$L=D_1^2+x_1^{2l}D_2^2+x_1^{2k}x_2^{2m}D_3^2+f(x)D_4^2 \qquad \text{in } R^4,$$

where l, k and m are positive integers and $f(x)=\exp(-1/|x_1|^\tau-1/|x_2|^\kappa)+\exp(-1/|x_1|^\delta-1/|x_2|^\sigma)$. Here $\tau=k+1+m(l+1)$, $0<\kappa<1$, $\delta>0$ and $\sigma>0$.

Theorem 3. (i) *Suppose that $l\geq k$. If $0<\delta<k+1$ and $0<\sigma<m+(k+1)/(l+1)$ then L is hypoelliptic in R^4 and moreover we have*

(10) $$\mathrm{WF}\,Lu=\mathrm{WF}\,u \qquad \text{for any } u\in\mathcal{D}'.$$

(ii) *Suppose that $k>l$. If $0<\delta<l+1$ and $0<\sigma<m+1$ then we have* (10).

Remark 4. In the case of (i), the assumption of Theorem 3 is optimal. That is, if either $\delta\geq k+1$ or $\sigma\geq m+(k+1)/(l+1)$ then L is not hypoelliptic in any neighborhood of the origin, (regardless of $l\geq k$). Furthermore, if $\sigma\geq m+1$ then we also get the non-hypoellipticity of L in any neighborhood of $\{x_2=0\}$. Those non-hypoellipticity results follow from the analogous method as in Theorem 1 of [2].

For the proof of the hypoellipticity of L we use the L^2 apriori estimate method as in [3] and [4]. The key point in the proof is to derive the following two estimates: For any $\varepsilon>0$ and any compact set K of R^4 there exists a constant $C_{\varepsilon,K}$ such that

(11) $$(x_1^{2k}x_2^{2m}(\log\Lambda)^2u,u)\leq\varepsilon(Lu,u)+C_{\varepsilon,K}\|u\|^2 \qquad \text{for } u\in C_0^\infty(K),$$

and furthermore

(12) $$(x_1^{2l}(\log \Lambda)^2u, u)\le \varepsilon(Lu, u)+C_{\varepsilon,K}\|u\|^2$$
$$\text{for } u\in C_0^\infty(K) \quad\text{with}\quad \operatorname{supp} u\cap\{x_2=0\}=\emptyset.$$

Here Λ denotes $(1+|D|^2)^{1/2}$. For a $M>0$ set

$$a(x,\xi)=\xi_1^2+x_1^{2l}\xi_2^2+\exp(-1/|x_1|^\delta-1/|x_2|^\sigma)M^2, \qquad W(x)=\varepsilon^{-1}x_1^{2k}x_2^{2m}(\log M)^2.$$

Then, by means of the microlocal analysis concentrated at $(0,(0,0,0,\pm 1))\in T^*R^4$, for the proof of the estimate (11) it suffices to show that the estimate (5) of Theorem 1 holds if $M>M_\varepsilon$ for a large M_ε. We shall check the assumption of (4) in the case of $l\ge k$. If K is a compact set of R^2 and if $\alpha=\{k+1+m(l+1)\}^{-1}$ and $\beta=(l+1)\alpha$ we set $\Omega_1=\{x\in K; |x_1|\le\rho_1(\log M)^{-\alpha}, |x_2|\le\rho_2(\log M)^{-\beta}\}$. Here ρ_j are small positives and in what follows we require that

(13) $$\rho_2\ll\rho_1\ll\varepsilon, \qquad \rho_1\ll 1/r^*,$$

where r^* denotes the modulus of the dilation of $(\cdot)^{**}$. Suppose that $B\in\mathcal{C}$ satisfies $\pi(B)\subset\Omega_1$. Then it follows from (13) that $\max_{\pi(B^{**})}W(x)\le\varepsilon^{-1}(\log M)^{2\alpha}$. Noting that $\xi_1^2\ge(4\rho_1)^{-2}(\log M)^{2\alpha}$ on a half of B, we get (4) in view of (13). If $\pi(B)$ is contained in $\{|x_1|\le\rho_1(\log M)^{-1/(k+1)}\}\cap K$ then we obtain (4) because we see that

$$\max_{\pi(B^{**})}W(x)\le\varepsilon^{-1}C_K(\log M)^{2/(k+1)} \quad\text{and}\quad \xi_1^2\ge(4\rho_1)^{-2}(\log M)^{2/(k+1)}$$

on a half of B. If B satisfies

(14) $$\pi(B)\subset\{|x_2|\le\rho_2(\log M)^{-\beta}\}\cap K,$$

(15) $$b\equiv\max_{\pi(B)}|x_1|\ge\rho_1(\log M)^{-\alpha},$$

then we see that

$$\max_{\pi(B^{**})}W(x)\le\varepsilon^{-1}(br^*)^{2k}(\log M)^{2-2\beta m} \quad\text{and}\quad x_1^{2l}\xi_2^2\ge 2^{-6}b^{2l}\rho_2^{-2}(\log M)^{2\beta}$$

on a quater of B. In view of $l\ge k$ and (15) we obtain (4) for this B. The assumption (4) for other $B\in\mathcal{C}$ is also obvious because we see that $\exp(-1/|x_1|^\delta-1/|x_2|^\sigma)M^2\ge M$ on

(16) $$\{|x_1|\ge(\rho_1/2)(\log M)^{-1/(k+1)}, |x_2|\ge(\rho_2/2)(\log M)^{-\beta}\}$$

if M is large enough that $(2/\rho_1)^\delta(\log M)^{\delta/(k+1)}$ and $(2/\rho_2)^\sigma(\log M)^{\beta\sigma}$ are less than $\log M^{1/2}$. In the case of $k>l$, the assumption (4) is checked by the same way as above if we replace β only in (14) and (16) by $(m+1)^{-1}$. The estimate (12) is also reduced to (5), by setting

$$a(x,\xi)=\xi_1^2+\exp(-1/|x_1|^\delta)M^2, \qquad W(x)=\varepsilon^{-1}x_1^{2l}(\log M)^2.$$

The way how estimates (11) and (12) lead us to the hypoellipticity of L will be shown elsewhere. The proofs of Theorems 1 and 2 will be also given elsewhere.

References

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