18. Correspondences for Hecke Rings and l-Adic Cohomology Groups on Smooth Compactifications of Siegel Modular Varieties

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Introduction. We report that the Hecke rings act on the l-adic cohomology groups of suitable non-singular projective toroidal compactifications of the higher dimensional modular varieties. We extend the fixed point theory of Lefschetz to the correspondences for the Hecke rings on those compactifications. We treat here the Siegel modular case. For details see Hatada [6], which will appear elsewhere.

§1. Let $g\geq 1$, $w\geq 0$, $j\geq 1$, $k\geq 1$, and $N\geq 3$ be rational integers. Let $\mathcal{R}$ denote a ring. Write

$M_{j,k}(\mathcal{R})=$the set of $j\times k$ matrices with coefficients in $\mathcal{R}$; $\mathcal{R}^j=M_{1,j}(\mathcal{R})$;

$1_k=$the $k\times k$ unit matrix $\in M_{k,k}(\boldsymbol{Z})$; $J_g=\begin{bmatrix}0 & -1_g\\ 1_g & 0\end{bmatrix}\in M_{2g,2g}(\boldsymbol{Z})$;

$\mathfrak{H}_g=$the Siegel upper half plane of degree g

$=\{Z\in M_{g,g}(\boldsymbol{C})\,|\,Z={}^tZ.\ \operatorname{Im} Z$ is positive definite.$\}$;

$\operatorname{Sp}(g,\boldsymbol{Z})=$the full symplectic modular group $\subset M_{2g,2g}(\boldsymbol{Z})$;

$\operatorname{GSp}^+(g,\boldsymbol{R})=\{\gamma\in \operatorname{GL}(2g,\boldsymbol{R})\,|\,{}^t\gamma J_g\gamma J_g^{-1}$ is a scalar matrix whose eigenvalue is positive.$\}$; $\operatorname{GSp}^+(g,\boldsymbol{Z})=\operatorname{GSp}^+(g,\boldsymbol{R})\cap M_{2g,2g}(\boldsymbol{Z})$;

$r(\alpha)=$the eigenvalue of ${}^t\alpha J_g\alpha J_g^{-1}$ for $\alpha\in\operatorname{GSp}^+(g,\boldsymbol{R})$;

$\operatorname{GSp}^+(g,\boldsymbol{R})\ltimes\boldsymbol{R}^{2gw}=$the semi-direct product of $\operatorname{GSp}^+(g,\boldsymbol{R})$ and $\boldsymbol{R}^{2gw}$ with $\boldsymbol{R}^{2gw}$ normal such that

$$(\alpha,\boldsymbol{m})\cdot(\beta,\boldsymbol{n})=\left(\alpha\cdot\beta,\ r(\beta)^{-1}\boldsymbol{m}\begin{bmatrix}\beta & & & 0\\ & \beta & & \\ & & \ddots & \\ 0 & & & \beta\end{bmatrix}+\boldsymbol{n}\right)$$

for all $\boldsymbol{m}$ and $\boldsymbol{n}\in\boldsymbol{R}^{2gw}$ and all α and $\beta\in\operatorname{GSp}^+(g,\boldsymbol{R})$. (In the right side the products are those for matrices.) We let $\operatorname{GSp}^+(g,\boldsymbol{R})\ltimes\boldsymbol{R}^{2gw}$ act on the complex analytic space $\mathfrak{H}_g\times\boldsymbol{C}^{gw}=\{(Z,\xi_1,\xi_2,\cdots,\xi_w)\,|\,Z\in\mathfrak{H}_g,\ \xi_j\in\boldsymbol{C}^g$ for any $j\in[1,w].\}$ to the left as follows. Write $\boldsymbol{m}=(\boldsymbol{m}_1,\boldsymbol{n}_1,\boldsymbol{m}_2,\boldsymbol{n}_2,\cdots,\boldsymbol{m}_w,\boldsymbol{n}_w)\in\boldsymbol{R}^{2gw}$ with $\boldsymbol{m}_j\in\boldsymbol{R}^g$ and $\boldsymbol{n}_j\in\boldsymbol{R}^g$ for any $j\in[1,w]$, and write $\alpha=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\operatorname{GSp}^+(g,\boldsymbol{R})$ partitioned into blocks on dimension $g\times g$. Then

$$\left(\begin{pmatrix}A & B\\ C & D\end{pmatrix},(\boldsymbol{m}_1,\boldsymbol{n}_1,\boldsymbol{m}_2,\boldsymbol{n}_2,\cdots,\boldsymbol{m}_w,\boldsymbol{n}_w)\right)(Z,\xi_1,\xi_2,\cdots,\xi_w)$$

$$=\left((AZ+B)(CZ+D)^{-1},\quad r\left(\begin{pmatrix}A & B\\ C & D\end{pmatrix}\right)\left(\xi_1+(\boldsymbol{m}_1,\boldsymbol{n}_1)\begin{pmatrix}Z\\ 1_g\end{pmatrix}\right)(CZ+D)^{-1},\right.$$

$$r\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)\left(\xi_2+(\boldsymbol{m}_2, \boldsymbol{n}_2)\begin{pmatrix} Z \\ 1_g \end{pmatrix}\right)(CZ+D)^{-1}, \cdots,$$
$$r\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)\left(\xi_w+(\boldsymbol{m}_w, \boldsymbol{n}_w)\begin{pmatrix} Z \\ 1_g \end{pmatrix}\right)(CZ+D)^{-1}\Bigg).$$

Let Γ be a congruence subgroup of $\mathrm{Sp}(g, \boldsymbol{Z})$ acting freely on the $\mathfrak{H}_g$. For simplicity assume that $\Gamma=\Gamma_g(N)=$the principal congruence subgroup of level N of $\mathrm{Sp}(g, \boldsymbol{Z})$. Let $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ denote the Hecke ring with respect to the group Γ and the monoid $\mathrm{GSp}^+(g, \boldsymbol{Z})$. (For the definition of the Hecke ring see e.g. Shimura [12, Chapter 3].) Let A_Γ be the universal principally polarized abelian variety with level N structure over the complex analytic quotient space $\Gamma\backslash\mathfrak{H}_g$. Let A_Γ^w be the w-fold fibred product of A_Γ over the $\Gamma\backslash\mathfrak{H}_g$, and let $E'\colon A_\Gamma^w\to\Gamma\backslash\mathfrak{H}_g$ be the canonical morphism. We may regard the complex analytic quotient space $(\Gamma\ltimes\boldsymbol{Z}^{2gw}\backslash\mathfrak{H}_g\times\boldsymbol{C}^{gw})$ as this A_Γ^w. Write A_Γ^0 for the $\Gamma\backslash\mathfrak{H}_g$. We consider simultaneous non-singular projective toroidal compactifications of A_Γ^w and A_Γ^0. Take a regular and projective $\mathrm{Sp}(g, \boldsymbol{Z})$-admissible family $\Sigma^{(0)}$ of polyhedral cone decompositions for the toroidal compactification of A_Γ^0. e.g. Take a suitable refinement of the second Voronoi decomposition. Then take its mixed cone decomposition. (cf. Namikawa [9] and [10].) By this we have a regular and projective $\mathrm{Sp}(g, \boldsymbol{Z})\ltimes\boldsymbol{Z}^{2gw}$-admissible family $\Sigma^{(1)}$ of polyhedral cone decompositions for the toroidal compactification of A_Γ^w. Then we get desired simultaneous compactifications $(A_\Gamma^w)^\sim$ and $(A_\Gamma^0)^\sim$ and a proper canonical morphism $E\colon (A_\Gamma^w)^\sim\to(A_\Gamma^0)^\sim$. For any element G of the family $\mathcal{G}$ defined in Hatada [5, Sec. 1] we compactify $G\backslash(\mathfrak{H}_g\times\boldsymbol{C}^{gw})$ toroidally with respect to this $\Sigma^{(1)}$. For simplicity write $M=(A_\Gamma^w)^\sim$ and ${}_0M=(A_\Gamma^0)^\sim$ from now on. Here $0\leq w\in\boldsymbol{Z}$. Let l be a prime number. Write

$Z_0(M, \boldsymbol{Z})=$the group of 0-cycles on M with $\boldsymbol{Z}$-coefficients;

$H_n(M, \boldsymbol{Z})=$the singular n-th homology group of M with $\boldsymbol{Z}$-coefficients;

$H^n(M, \boldsymbol{Q}_l)=$the n-th l-adic cohomology group of M (cf. SGA 4 [1]);

$H^n({}_0M, (R^mE_*\boldsymbol{Q}_l)^{\otimes j})=$the n-th l-adic cohomology group of ${}_0M$ with coefficients in the j-fold tensor product of the m-th l-adic direct image sheaf $R^mE_*\boldsymbol{Q}_l$ (cf. SGA 4 [1]).

Lemma 1. *There is a ring homomorphism* $h_0\colon HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))\to \mathrm{End}_{\boldsymbol{Z}}Z_0(M, \boldsymbol{Z})$. *In using the notations in Hatada* [5, Sec. 2, (2.2.1) *and* (DEF. 1)], *it is given by*

$$(h_0(\Gamma\alpha\Gamma))(Q)=\sum_{i=1}^{\mu}[\pi]\circ(\alpha_i, \mathbf{0})^\sim\circ\pi_i(Q')$$

where Q is a closed point of M; Q' is a closed point of (Domain of $\pi^{(i)}\circ\pi_i$) such that $Q=\pi^{(i)}\circ\pi_i(Q')$; $\alpha\in\mathrm{GSp}^+(g, \boldsymbol{Z})$ and $\Gamma\alpha\Gamma=\cup_{i=1}^{\mu}\Gamma\alpha_i$ (disjoint).

Let $\alpha\in\mathrm{GSp}^+(g, \boldsymbol{Z})$. Let $|(h_0(\Gamma\alpha\Gamma))(Q)|$ denote the support $(\cup_{i=1}^{\mu}[\pi]\circ(\alpha_i, \mathbf{0})^\sim\circ\pi_i(Q'))$ of the 0-cycle $(h_0(\Gamma\alpha\Gamma))(Q)$. Define $\mathcal{Z}_M(\Gamma\alpha\Gamma)$ to be the subscheme: $\cup_{Q\in M}\{Q\}\times|(h_0(\Gamma\alpha\Gamma))(Q)|$ of the product variety $M\times M$. For an element $\mathcal{A}=\sum_i a_i\Gamma\alpha_i\Gamma$ of $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ where each $a_i\neq0$ and $\Gamma\alpha_i\Gamma\neq\Gamma\alpha_{i'}\Gamma$ if $i\neq i'$, define $\mathcal{Z}_M(\mathcal{A})$ to be $\cup_i\mathcal{Z}_M(\Gamma\alpha_i\Gamma)$. Recall

Theorem A (Theorem 1 in Hatada [5]). *The Hecke ring* $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ *acts on* $H_n(M, \boldsymbol{Z})$, i.e., *there is a natural ring homomorphism* $f_n: HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \to \mathrm{End}_{\boldsymbol{Z}} H_n(M, \boldsymbol{Z})$ *for each integer* $n \geq 0$.

For any $u \in \mathrm{End}_{\boldsymbol{Z}} H_n(M, \boldsymbol{Z})$ let $u \otimes_{\boldsymbol{Z}} \mathrm{id.}$ denote the element of $\mathrm{End}_{\boldsymbol{Q}} H_n(M, \boldsymbol{Q})$ through the isomorphism $H_n(M, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \cong H_n(M, \boldsymbol{Q})$. Let ${}^t(u \otimes_{\boldsymbol{Z}} \mathrm{id.})$ denote the transposed linear map $\in \mathrm{End}_{\boldsymbol{Q}} H^n(M, \boldsymbol{Q})$ with respect to the complete duality of the Kronecker index $H^n(M, \boldsymbol{Q}) \times H_n(M, \boldsymbol{Q}) \to \boldsymbol{Q}$. Write $d = \dim_{\boldsymbol{C}} M$. Let $D_n: H^{2d-n}(M, \boldsymbol{Q}) \to H_n(M, \boldsymbol{Q})$ denote the $\boldsymbol{Q}$-linear isomorphism by the Poincaré duality for each $n \geq 0$. Define *Coincidence Number* $L_M(\mathcal{A}_1, \mathcal{A}_2)$ on M for elements $\mathcal{A}_1$ and $\mathcal{A}_2 \in HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ by

$$L_M(\mathcal{A}_1, \mathcal{A}_2) = \sum_{n=0}^{2d} (-1)^n \operatorname{Tr} \{D_n \circ {}^t(f_{2d-n}(\mathcal{A}_2) \otimes_{\boldsymbol{Z}} \mathrm{id.}) \circ D_n^{-1} \circ (f_n(\mathcal{A}_1) \otimes_{\boldsymbol{Z}} \mathrm{id.})\}.$$

We obtain

Theorem 1. *There exists a certain natural* $\boldsymbol{Z}$*-bilinear map* F_n: $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \times HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \to \mathrm{Hom}_{\boldsymbol{Z}}(H_n(M, \boldsymbol{Z}), H_n(M \times M, \boldsymbol{Z}))$ *for each integer* $n \geq 0$. (For details on the definition and the construction of $\{F_n\}_{n \geq 0}$ see Hatada [6].)

Write $\Delta(M) = \{(x, y) \in M \times M \mid x = y\}$. Let $\iota: (M \times M, \phi) \to (M \times M, M \times M - \Delta(M))$ be the inclusion map for pairs of topological spaces. Let ι_{*n} denote the induced $\boldsymbol{Z}$-linear map: $H_n(M \times M, \boldsymbol{Z}) \to H_n(M \times M, M \times M - \Delta(M), \boldsymbol{Z})$ for each $n \geq 0$. Let z denote the fundamental class of M, and let U denote the Thom class of M. One has the isomorphism: $H_{2d}(M \times M, M \times M - \Delta(M), \boldsymbol{Z}) \cong \boldsymbol{Z}$ given by $\zeta \mapsto \langle U, \zeta \rangle_{2d}$ under the Kronecker index. Define *Coincidence Index* $I_M(\mathcal{A}_1, \mathcal{A}_2)$ on M for elements $\mathcal{A}_1$ and $\mathcal{A}_2 \in HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$ to be the integer corresponding to the class

$$\iota_{*2d} \circ (F_{2d}((\mathcal{A}_1, \mathcal{A}_2))(z) \in H_{2d}(M \times M, M \times M - \Delta(M), \boldsymbol{Z})$$

under the above isomorphism.

We obtain the following theorems.

Theorem 2. $L_M(\mathcal{A}_1, \mathcal{A}_2) = I_M(\mathcal{A}_1, \mathcal{A}_2)$ *for all* $\mathcal{A}_1$ *and* $\mathcal{A}_2 \in HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. (We call this Coincidence Theorem.)

Theorem 3. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2 \in HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z}))$. *If* $L_M(\mathcal{A}_1, \mathcal{A}_2) \neq 0$, *then* $\mathcal{Z}_M(\mathcal{A}_1) \cap \mathcal{Z}_M(\mathcal{A}_2) \neq \phi$. (We call this Fixed Point Theorem.)

Theorem 4. *The Hecke ring* $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}_l$ *over* $\boldsymbol{Q}_l$ *acts on* $H^n(M, \boldsymbol{Q}_l)$ *as an anti-ring homomorphism,* i.e., *there exists a natural anti-ring homomorphism* $f^{(n)}: HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}_l \to \mathrm{End}_{\boldsymbol{Q}_l} H^n(M, \boldsymbol{Q}_l)$ *for each integer* $n \geq 0$.

Theorem 5. *The Hecke ring* $HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}_l$ *over* $\boldsymbol{Q}_l$ *acts on* $H^n({}_0M, (R^m E_* \boldsymbol{Q}_l)^{\otimes j})$ *as an anti-ring homomorphism,* i.e., *there exists a natural anti-ring homomorphism* $\mathfrak{f}^{(n,m,j)}: HR(\Gamma, \mathrm{GSp}^+(g, \boldsymbol{Z})) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}_l \to \mathrm{End}_{\boldsymbol{Q}_l} H^n({}_0M, (R^m E_* \boldsymbol{Q}_l)^{\otimes j})$ *for each integers* $n \geq 0$, $m \geq 0$ *and* $j \geq 1$.

Remark 1. The above Theorems 4 and 5 also hold true for the natural Siegel modular schemes over $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{F}}_p$ with $p \nmid N$ instead of those schemes over $\boldsymbol{C}$ treated in this note. Here $\Gamma = \Gamma_g(N) \subset \mathrm{Sp}(g, \boldsymbol{Z})$ and $l \neq p$.

Remark 2. Let $\alpha \in \mathrm{GSp}^+(g, \boldsymbol{Z})$. Write $\mathfrak{D}=\mathfrak{H}_g \times \boldsymbol{C}^{gw}$. Set $X=$the complex analytic quotient space $((\Gamma \ltimes \boldsymbol{Z}^{2gw}) \cap ((\alpha, \mathbf{0})^{-1}(\Gamma \ltimes \boldsymbol{Z}^{2gw})(\alpha, \mathbf{0}))) \backslash \mathfrak{D}$. Let $v_1: X \rightarrow (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})$ and $v_2: X \rightarrow ((\alpha, \mathbf{0})^{-1}(\Gamma \ltimes \boldsymbol{Z}^{2gw})(\alpha, \mathbf{0}) \backslash \mathfrak{D})$ be the canonical maps. Let $[(\alpha, \mathbf{0})]$ be the map: $((\alpha, \mathbf{0})^{-1}(\Gamma \ltimes \boldsymbol{Z}^{2gw})(\alpha, \mathbf{0}) \backslash \mathfrak{D}) \rightarrow (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})$ induced from the map $(\alpha, \mathbf{0}): \mathfrak{D} \rightarrow \mathfrak{D}$. Write $v_3=[(\alpha, \mathbf{0})] \circ v_2$. Consider the graph of $v_3 \circ {}^t v_1$, for which we write $\mathrm{GR}(v_3 \circ {}^t v_1)$, in the product $(\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D}) \times (\Gamma \ltimes \boldsymbol{Z}^{2gw} \backslash \mathfrak{D})$. Then the $\mathcal{Z}_M(\Gamma \alpha \Gamma)$ coincides with the closure of $\mathrm{GR}(v_3 \circ {}^t v_1)$ in $M \times M$ with respect to the usual complex topology and also with respect to the Zariski topology.

Note added in proof. Since M is a complex projective manifold, it is orientable. Let s be an orientation of M. The fundamental class z (resp. the Thom class U) of M used in this note is the one attached to s.

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