93. Matrix Prime Number Theorems

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0. The purpose of this paper is to show how some information on prime numbers in very short intervals, of type $[x, x+x^{\varepsilon}]$ with $\varepsilon>0$ arbitrary, may be derived by classical methods of analytic number theory. We have not attempted in this note to give the sharpest result possible in this direction. For comparison, the known applications for zero-density results give

$$\pi(x+x^{\delta})-\pi(x)\sim x^{\delta}/\log x \quad (\delta>7/12)$$

and a corresponding result for almost all intervals for $\delta>1/6$ (Huxley [1], [3], p. 19).

1. In this section we derive the main estimate in our work. Deductions from it will be given in section 2.

We shall need from the theory the following known results. We use the notation $\rho=\beta+i\gamma$ for the real and imaginary parts of a non-trivial zero of Riemann zeta-function.

(A) The number of ρ with $T\leqslant|\gamma|<T+1$ is $O(\log T)$, where multiplicities are counted.

This is Theorem 9.2 in Titchmarsh [4].

(B) We have $1-\beta\gg(\log\gamma)^{-2/3}(\log\log\gamma)^{-1/3}$.

This is the Vinogradov-Korobov bound ([4], p. 135).

(C) The explicit formula with remainder of prime number theory may be taken as

$$y-\psi(y)=\sum_{|\gamma|<T}\frac{y^{\rho}}{\rho}+O(\log y+yT^{-1}(\log T)^{2})$$

provided T, y are greater than 1 and y is bounded by a fixed power of T.

This follows from Theorem 3.8 of Patterson [2].

Notation. *Let η be fixed with $0<\eta<1$, let $X>1$ and $N=X^{\eta}$. Define $\alpha=X^{1/N}$.*

Theorem. *Suppose N is an integer. Define $f(\theta)$ to be*

$$\sum_{j=1}^{N}\Big(\sum_{\alpha^{j-1}<m\leqslant\alpha^{j}}\Lambda(m)-\alpha^{j-1}(\alpha-1)\Big)e^{ij\theta}.$$

Then $f(\theta)$ is $o(X)$ (with constant independent of θ).

Remarks. The integrality of N is assumed to simplify the notation; sums over m in $(\alpha^{-r}X, \alpha^{-r+1}X]$ may be dealt with in the same way in general. The proof shows that the bound may be taken as

$$O(X\exp(-A(\log X)^{1/3}(\log\log X)^{-1/3})).$$

We prove first the following

Main lemma. *With notation as before, let* $\varepsilon>0$. *Then*

$$\Sigma=\sum_{0<\gamma<N^{1+\varepsilon}} g(\gamma)h(\gamma,\theta)\ll_{\varepsilon}(\log X)^4,$$

where $g(\gamma)$ *is* $\log X/N$ *for* $0<\gamma\leqslant N/(\log X)^2$ *and* γ^{-1} *for* $\gamma>N/(\log X)^2$, *and* $h(\gamma,\theta)$ *is* $\min(N,\|(2\pi)^{-1}(\gamma\log X\cdot N^{-1}+\theta)\|^{-1})$. *Here* $\|t\|$ *denotes the distance of* t *to the nearest integer*; *and the minimum has the natural interpretation as* N *when the second member is not defined.*

Remark. Here and in all subsequent sums it is assumed that the γ are counted with multiplicity.

Proof of the Lemma. Firstly Σ is a sum $\Sigma_1+\Sigma_2$ where Σ_1 relates to the range up to $N/(\log X)^2$ and Σ_2 to the rest. We shall prove

(i) $\Sigma_1\ll(\log X)^3$

and

(ii) $\Sigma_2\ll_{\varepsilon}(\log X)^4$;

this is enough for the Main lemma.

Subproof of (i). We have $\Sigma_1=\log X\cdot N^{-1}\sum_{0<\gamma\leqslant N/(\log X)^2} h(\gamma,\theta)$. Consider the sums $\Sigma_{(k)}$ defined as Σ_1 but with the extra condition that $N^{-1}h(\gamma,\theta)$ lies in $[k,k+1)$ for $k=0,1,\cdots,M$ and $M\leqslant N$. We prove

(*) $\Sigma_{(k)}\ll(\log X)^2\min(1,k^{-1})$

for all k (with the natural convention for $k=0$) and then (i) follows by the harmonic series summation.

For (*) we note that for any t and t' the inequality

$$k/N\leqslant\left\|\frac{\log Xt+t'}{2\pi N}\right\|<(k+1)/N$$

defines (for fixed t') at most 2 intervals in t, for $0\leqslant t<2\pi N/\log X$ of length at most $2\pi/\log X$. By (A) a total of at most $O(\log X)$ ordinates γ is involved in $\Sigma_{(k)}$, and this is enough.

Subproof of (ii). Firstly split the range in Σ_2 into a number whose lengths are in geometric progression by powers of 2; it is enough, since $\log_2 N^{1+\varepsilon}\ll_{\varepsilon}\log X$, to prove that

(**) $\Sigma'_a=\sum_{T(a)\leqslant\gamma<T(a+1)}\gamma^{-1}h(\gamma,\theta)\ll(\log X)^3$

where $T(a)$ is $2^aN/(\log X)^2$. Here the factor γ^{-1} may be replaced by $T(a)^{-1}$. By further splitting into sums $\Sigma'_{a,(k)}$ according to the value of $N^{-1}h(\gamma,\theta)$ exactly as in (i), we are reduced to proving

(***) $\Sigma'_{a,(k)}\ll(\log X)^2\min(1,k^{-1})$.

The argument for (***) is as before in (i), using (A), but this time the number of intervals is bounded by $T(a)/(2\pi N/\log X)$. Thus the factors $T(a)$ cancel and the result is entirely analogous to (*). This completes the proof of (ii), and with it the Main lemma.

Proof of the Theorem. From (C), noting that the coefficient of $e^{ij\theta}$ is $\psi(\alpha^j)-\alpha^j-\psi(\alpha^{j-1})+\alpha^{j-1}$, we see that $f(\theta)$ may be replaced by

$$f_1(\theta)=-\sum_j\sum_{|\gamma|<N^{1+\varepsilon}}\rho^{-1}\alpha^{j\rho}(1-\alpha^{-\rho})e^{ij\theta}$$

with error which is $O(N(\log X+N^{-1-\varepsilon}X(\log X)^2))$, for $\varepsilon>0$, and so $o(X)$

since $\eta>0$. (Naturally ε could be a suitable function of X here.) Now reversing the summation in $-f_1(\theta)$ and summing geometric progressions with ratio $\alpha^{-\rho}e^{-i\theta}$, we find we must establish

$$(\dagger)\quad \sum_{|\gamma|<N^{1+\varepsilon}} \rho^{-1}x^{\rho}(1-\alpha^{-\rho})e^{iN\theta}(x^{-\rho}e^{-iN\theta})/(\alpha^{-\rho}e^{-i\theta}-1)=o(X).$$

To bring this to a form in which the Main lemma applies, we first note that we may take $\gamma>0$ by splitting conjugate pairs and changing θ for $-\theta$, so that ($\dagger$) follows from

$$(\dagger\dagger)\quad \sum_{0<\gamma<N^{1+\varepsilon}} |\rho^{-1}x^{\rho}(1-\alpha^{-\rho})(x^{-\rho}e^{-iN\theta}-1)(\alpha^{-\rho}e^{-i\theta}-1)^{-1}|=o(X).$$

Now for $r\leqslant N/(\log X)^2$ we have

$$\rho^{-1}(1-\alpha^{-\rho})\ll\log\alpha=\log X/N.$$

Otherwise we have $|\rho|>\gamma$, so that in the sum in ($\dagger\dagger$) we get, after bounding $1-\alpha^{-\rho}$ in the upper range of γ, a sum of the shape in the Main lemma but with $h(\gamma,\theta)$ replaced by $X^{\beta}h'(\gamma,\theta)$ where

$$h'(\gamma,\theta)=|x^{-\rho}e^{-iN\theta}-1|/|\alpha^{-\rho}e^{-i\theta}-1|.$$

Here we use the natural convention that h' is defined by continuity is the case the denominator vanishes.

We bound X^{β} by (B), choosing for example $\varepsilon>0$ so that all γ are bounded by X. This gives at least $X^{\beta}=O(X(\log X)^{-A})$ for all $A>0$; thus ($\dagger\dagger$) and so the theorem follows from the Main lemma and

($\dagger\dagger\dagger$) $h'(\gamma,\theta)\ll h(\gamma,\theta)$.

Firstly this is true when $\mathrm{Re}(\alpha^{-\rho}e^{-i\theta})$ is negative, for then the left hand side is absolutely bounded above and the right hand side is $\gg 1$. In the other case we have

$$|\alpha^{-\rho}e^{-i\theta}-1|\geqslant|w-1|$$

where $w=\exp(i(-\gamma\log X/N-\theta))$—geometrically this says that the radius from O to w lies wholly outside the circle centre 1 and radius $|w-1|$. We write

$$|w-1|=2|\sin t|\qquad\text{where } e^{2it}=w,$$

and then use the fact that $|\sin t|$ lies between $2\pi^{-1}|t|$ and $|t|$ for $|t|\leqslant\pi/2$. The required cases of ($\dagger\dagger\dagger$), including the cases where the minimum on either side is N and the degenerate case $w=1$, follow at once. This completes the proof of the Theorem.

2. Since the theorem provides a uniform non-trivial bound for $f(\theta)$, there are corresponding bounds, which give this paper its title, for associated operators and certain integrals.

For example

$$\int_0^{2\pi} e^{i(j-k)\theta}(f(\theta)+f(-\theta))\,d\theta$$

has a uniform (in j,k) bound $o(X)$, and so bounds can be given for the L^2-norm (which is the spectral radius max $|\lambda|$ taken over eigenvalues) for a family of symmetric Toeplitz matrices $(c_{j-k})_{jk}$ formed from the coefficients of f. This is what constitutes a "matrix prime number theorem" in our terminology. To convert such a statement into a positivity result, of the expected kind (after Weil [5]), is easily done, since adding $o(X)I$ to these matrices makes them positive definite.

For a small modification we can get Hankel matrices (c_{j+k}) and similar bounds. From the proof it is clear that $\beta \leqslant \sigma$, i.e. partial progress to the Riemann Hypothesis, would immediately give $O(X^{\sigma})$. In the same way it seems that the nuclear norm problem of estimating $\Sigma|\lambda|$ is related to the question of primes in short intervals as in the work of Huxley quoted in section 0; and the Hilbert-Schmidt norm problem of estimating $\Sigma|\lambda|^2$ is similarly related to the problem of primes in almost all short intervals also mentioned there.

Finally a different method may be used to estimate the spectral radius for matrices $(|c_{j-k}|)$ as above, for $\eta > 1/6$.

References

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