8. Minimal Quasi-ideals in Abstract Affine Near-rings

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1. Introduction. In his paper [3], Stewart answered Problem 6.1 in [2] in the affirmative, proving that a quasi-ideal of a ring is minimal if and only if any two of its non-zero elements generate the same left ideal and the same right ideal of the ring.

Our aim is to generalize the above result to a class of abstract affine near-rings. An example is given to show that the result does not hold for arbitrary near-rings.

2. Left ideals, right ideals and quasi-ideals. Let N be a near-ring, which always means right one throughout this note. If A and B are two non-empty subsets of N, then AB denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$, and $A * B$ denotes the set of all finite sums of the form $\sum (a_k(a'_k + b_k) - a_k a'_k)$ with $a_k, a'_k \in A$, $b_k \in B$.

A *left ideal* of N is a normal subgroup L of $(N, +)$ such that $N * L \subseteq L$, and a *right ideal* of N is a normal subgroup R of $(N, +)$ such that $RN \subseteq R$. A *quasi-ideal* of N is a subgroup Q of $(N, +)$ such that $N * Q \cap NQ \cap QN \subseteq Q$. A non-zero quasi-ideal Q is *minimal* if the only quasi-ideals of N contained in Q are $\{0\}$ and Q. Left ideals and right ideals are quasi-ideals, and the intersection of a family of quasi-ideals is again a quasi-ideal.

A near-ring N is called an *abstract affine near-ring* if N is abelian and $N_0 = N_d$, where N_0 is the zero-symmetric part of N and N_d is the set of all distributive elements of N.

These definitions lead immediately to

Lemma. *Let N be an abstract affine near-ring.*

(a) *A subgroup L of $(N, +)$ is a left ideal of N if and only if $N_0 L \subseteq L$.*

(b) *If P is a subgroup of $(N, +)$, then $N_0 P$ is a left ideal of N and PN is a right ideal of N.*

(c) *A subgroup Q of $(N, +)$ is a quasi-ideal of N if and only if $N_0 Q \cap QN \subseteq Q$.*

3. Main result. If x is an element of a near-ring N, $(x)_l$ (respectively, $(x)_r$) denotes the left (respectively, right) ideal of N generated by x. Now we are ready to state the main result of this note.

Theorem. *A quasi-ideal Q of an abstract affine near-ring N is minimal if and only if any two of its non-zero elements generate the same left ideal and the same right ideal of N.*

Proof. Suppose that Q is a minimal quasi-ideal of an abstract affine near-ring N. Let x be a non-zero element of Q. Then $(x)_l \cap Q$ is a non-zero quasi-ideal contained in Q. So $(x)_l \cap Q = Q$. Thus $Q \subseteq (x)_l$. Let y be a non-zero element of Q. Then $y \in (x)_l$. So $(y)_l \subseteq (x)_l$. Similarly, we have $(x)_l \subseteq (y)_l$ and so $(x)_l = (y)_l$. Thus any two non-zero elements of Q generate the same left (and similarly the same right) ideal of N.

Conversely, suppose that any two non-zero elements of a quasi-ideal Q of an abstract affine near-ring N generate the same left ideal and the same right ideal of N. Let P be a non-zero quasi-ideal of N such that $P \subseteq Q$.

First assume that $N_0P \cap Q \neq \{0\}$ and that $PN \cap Q \neq \{0\}$. Let p be a non-zero element of $N_0P \cap Q$ and q be a non-zero element of $PN \cap Q$. Then, for any non-zero element x of Q, by Lemma (b), we have $x \in (x)_l = (p)_l \subseteq N_0P$ and $x \in (x)_r = (q)_r \subseteq PN$. Thus, by Lemma (c), $x \in N_0P \cap PN \subseteq P$. So $Q = P$.

Now assume that $N_0P \cap Q = \{0\}$. Let y be a non-zero element of P. Then, for any non-zero element x of Q, we have $x \in (x)_l = (y)_l$. Thus, by Lemma (a), $x = my + ny$ for some integer m and some element n of N_0. It follows that $ny = x - my \in N_0P \cap Q = \{0\}$. So $x = my \in P$, and $Q = P$.

A similar argument shows that $Q = P$, if $PN \cap Q = \{0\}$. Thus Q is minimal.

4. Remarks. For arbitrary near-ring N, it is true that any two non-zero elements of a quasi-ideal Q of N generate the same left ideal and the same right ideal of N, if Q is minimal. But the converse does not hold in general.

Take a near-ring on the alternating group $A_4 = \{0, 1, 2, \cdots, 11\}$ with the trivial multiplication $ab = 0$ for all $a, b \in A_4$. Then the normal subgroup $I = \{0, 1, 2, 3\}$ of $(A_4, +)$ is a quasi-ideal of A_4, and $(k)_l = I = (k)_r$ for $k = 1, 2, 3$. But I is not minimal, since it contains a quasi-ideal $\{0, 1\}$.

References

[1] G. Pilz: Near-rings. North-Holland, Amsterdam (1983).

[2] O. Steinfeld: Quasi-ideals in rings and semigroups. Akadémiai Kiadō, Budapest (1978).

[3] P. N. Stewart: Quasi-ideals in rings. Acta Math. Acad. Sci. Hungar., **38**, 231–235 (1981).