92. *Note on Heinz's Inequality*

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The operator monotone functions are completely characterized by K. Löwner. But the proof is by no means short or elementary. For instance, it is not at all obvious that $f(t)=t^{1/2}$ is operator monotone. And in fact it was discovered by E. Heinz in 1951 that $f(t)=t^{\nu}$ was operator monotone for $\nu \varepsilon [0,1]$. In the following year T. Katô gave a shorter proof of the another Heinz's inequality.

In this note, it will be proved that Löwner's special case, Heinz's inequality, Heinz-Katô type inequality and the recent Chan-Kwong's result are all equivalent.

We use capital letters $A, B, \cdots$ to denote the bounded linear operators on the Hilbert space $\mathcal{H}$.

Theorem. *The following results* (i)–(iv) *are equivalent.*

(i) (K. Löwner) *If* $A \geqq B \geqq 0$, *then* $A^{1/2} \geqq B^{1/2}$.

(ii) (N. N. Chan–M. K. Kwong) *If* $A \geqq B \geqq 0$, $C \geqq D \geqq 0$, $AC=CA$ *and* $BD=DB$, *then* $A^{1/2}C^{1/2} \geqq B^{1/2}D^{1/2}$.

(iii) (E. Heinz) *If* $A \geqq B \geqq 0$, *then* $A^{\nu} \geqq B^{\nu}$ *for all* $\nu \varepsilon [0,1]$.

(iv) (E. Heinz–T. Katô) *If* $A \geqq 0$, $B \geqq 0$, $\|Qx\| \leqq \|Ax\|$, $\|Q^*y\| \leqq \|By\|$ *for all* $x, y \varepsilon \mathcal{H}$, *then* $|\langle Qx, y\rangle| \leqq \|A^{\nu}x\| \|B^{1-\nu}y\|$ *for all* $\nu \varepsilon [0,1]$.

To prove Theorem we need the following Lemmas.

Lemma 1. *If* $A \geqq B > 0$, *then* $A^{-1} \leqq B^{-1}$.

Proof. If $A \geqq B > 0$, then $B^{-1/2}AB^{-1/2} \geqq I$ and $B^{1/2}A^{-1}B^{1/2} \leqq I$ and hence $A^{-1} \leqq B^{-1}$.

Lemma 2. *If* (i) *of Theorem is fulfilled and if* $E \geqq F > 0$, $X \geqq 0$, $Y \geqq 0$ *and* $XFX \geqq YEY$, *then* $X \geqq Y$.

Proof. Since $XEX \geqq XFX \geqq YEY$ by the assumptions, $E^{1/2}XEXE^{1/2} \geqq E^{1/2}YEYE^{1/2}$ and $E^{1/2}XE^{1/2} \geqq E^{1/2}YE^{1/2}$ by (i) and hence $X \geqq Y$.

Proof of Theorem. (i) *implies* (ii); For any $\varepsilon > 0$, let $A_{\varepsilon}=A+\varepsilon I$, then $A_{\varepsilon} \geqq B_{\varepsilon} \geqq \varepsilon I > 0$ and $B_{\varepsilon}^{-1} \geqq A_{\varepsilon}^{-1} > 0$ by Lemma 1. Let $X=(A_{\varepsilon}C)^{1/2}$ and $Y=(B_{\varepsilon}D)^{1/2}$, then $X \geqq 0$, $Y \geqq 0$ and $XA_{\varepsilon}^{-1}X=(A_{\varepsilon}C)^{1/2}A_{\varepsilon}^{-1}(A_{\varepsilon}C)^{1/2}=C \geqq D=YB_{\varepsilon}^{-1}Y$ and hence $X \geqq Y$ by Lemma 2. This implies that $A^{1/2}C^{1/2}=B^{1/2}D^{1/2}$.

(ii) *implies* (iii); If $A^{\alpha} \geqq B^{\alpha} \geqq 0$, $A^{\beta} \geqq B^{\beta} \geqq 0$ for $\alpha, \beta \varepsilon [0,1]$, then $A^{(\alpha+\beta)/2} \geqq B^{(\alpha+\beta)/2}$ by (ii) and hence $A^{\nu} \geqq B^{\nu}$ for all $\nu \varepsilon [0,1]$.

(iii) *implies* (iv); Let $Q=V|Q|=|Q^*|V$ be the polar decomposition of Q, then, for any $x, y \varepsilon \mathcal{H}$, $\||Q|x\|=\|Qx\| \leqq \|Ax\|$, $\||Q^*|y\|=\|Q^*y\| \leqq \|By\|$ and $\||Q|^{\nu}x\| \leqq \|A^{\nu}x\|$, $\||Q^*|^{1-\nu}y\| \leqq \|B^{1-\nu}y\|$ for all $\nu \varepsilon [0,1]$ by (iii) and hence $|\langle Qx, y\rangle|$

$=|\langle V|Q|x, y\rangle|=|\langle |Q|^{\nu}x, |Q|^{1-\nu}V^*y\rangle|=|\langle |Q|^{\nu}x, V^*|Q^*|^{1-\nu}y\rangle|\leqq \||Q|^{\nu}x\|\||Q^*|^{1-\nu}y\|\leqq \|A^{\nu}x\|\|B^{1-\nu}y\|$.

(iv) *implies* (i); (J. Dixmier) Since $\|(B^{1/2})^*x\|^2=\|B^{1/2}x\|^2=\langle Bx, x\rangle\leqq \langle Ax, x\rangle=\|A^{1/2}x\|^2$, let $Q=B^{1/2}$, $\nu=1/2$ and $x=y$ in (iv), then $\langle B^{1/2}x, x\rangle= |\langle Qx, x\rangle|\leqq\|A^{1/4}x\|^2=\langle A^{1/2}x, x\rangle$ for all $x\in\mathcal{H}$ and $B^{1/2}\leqq A^{1/2}$.

Remark. To prove the Heinz's inequality, by Theorem, we have only to show (i) is fulfilled. And it seems that the following proof is most elegant and simple: (G. K. Pedersen) For any $\varepsilon>0$, let $A_\varepsilon=A+\varepsilon I$, then $A_\varepsilon> B\geqq 0$ and $A_\varepsilon^{-1/2}BA_\varepsilon^{-1/2}<I$ and hence $\|B^{1/2}A_\varepsilon^{-1/2}\|<1$. Since

$$\begin{aligned}|\sigma(A_\varepsilon^{-1/4}B^{1/2}A_\varepsilon^{-1/4})|&=\lim_{n\to\infty}\|(A_\varepsilon^{-1/4}B^{1/2}A_\varepsilon^{-1/4})^n\|^{1/n}\\ &=\lim_{n\to\infty}\|A_\varepsilon^{-1/4}(B^{1/2}A_\varepsilon^{-1/4}A_\varepsilon^{-1/4})^{n-1}B^{1/2}A_\varepsilon^{-1/4}\|^{1/n}\\ &\leqq\lim_{n\to\infty}\|A_\varepsilon^{-1/4}\|^{1/n}\{\|(B^{1/2}A_\varepsilon^{-1/2})^{n-1}\|^{1/(n-1)}\}^{(n-1)/n}\|B^{1/2}A_\varepsilon^{-1/4}\|^{1/n}\\ &=|\sigma(B^{1/2}A_\varepsilon^{-1/2})|\leqq\|B^{1/2}A_\varepsilon^{-1/2}\|<1\end{aligned}$$

where $|\sigma(A)|$ denotes the spectral radius of A, $A_\varepsilon^{-1/4}B^{1/2}A_\varepsilon^{-1/4}<I$ and $B^{1/2}<A_\varepsilon^{1/2}$ and hence $B^{1/2}\leqq A^{1/2}$.

References

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