74. Some Generalizations of Chebyshev's Conjecture

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1988)

§1. **Statement of the results and the conjecture.** Chebyshev [2] asserted, in 1853, that

$$\lim_{x\to+0}\sum_{p>2}(-1)^{(p-1)/2}e^{-xp}=-\infty,$$

where p runs over the odd prime numbers, although the proof has never been published. In fact, in 1917, Hardy-Littlewood [3] and Landau [5] have shown that the above statement is equivalent to the Generalized Riemann Hypothesis (G.R.H.) for the Dirichlet L-function $L(s,\chi)$ with the non-principal character χ mod 4. Later, Knapowski and Turan [4] have extensively studied this subject, and proved among others that the following statement is equivalent to G.R.H. for $L(s,\chi)$

$$\lim_{x\to+0}\sum_{p>2}(-1)^{(p-1)/2}\log p\; e^{-\log^2(px)}=-\infty.$$

The purpose of the present article is to give a generalization of Chebyshev's conjecture and prove its equivalence to G.R.H. for $L(s,\chi)$ for some special cases.

To state our theorem we define the function $\xi(x,k)$ by

$$\Gamma^k(s)=\int_0^\infty x^{s-1}\xi(x,k)dx,$$

where $\Gamma(s)$ is the Γ-function and k is an integer $\geqq 1$. $\xi(x,1)=e^{-x}$ and $\xi(x,2)=2K_0(2\sqrt{x})$ with the Bessel function $K_0(x)$. We shall prove the following theorems.

Theorem 1. *Suppose that $0<\alpha<\alpha_0$, where α_0 may be >4. Then the statement that*

$$\lim_{x\to+0}\sum_{p>2}(-1)^{(p-1)/2}e^{-(xp)^\alpha}=-\infty$$

is equivalent to G.R.H. *for $L(s,\chi)$.*

Theorem 2. *The statement that*

$$\lim_{x\to+0}\sum_{p>2}(-1)^{(p-1)/2}\cdot\log p\cdot\xi(xp,2)=-\infty$$

is equivalent to G.R.H. *for $L(s,\chi)$.*

We may state our generalization of Chebyshev's conjecture as follows.

Conjecture. (i) *For any positive α,*

$$\lim_{x\to+0}\sum_{p>2}(-1)^{(p-1)/2}e^{-(xp)^\alpha}=-\infty.$$

(ii) *For any integer $k\geqq 1$,*

$$\lim_{x\to+0}\sum_{p>2}(-1)^{(p-1)/2}\xi(xp,k)=-\infty.$$

§2. **Proof of Theorem 2.** We denote $\xi(x,2)$ by $f(x)$ and $\Gamma^2(s)$ by $F(s)$, for simplicity. We use the well known properties of $L(s,\chi)$, $f(x)$ and

$F(s)$ without mentioning the references.

$$-\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s)\frac{L'}{L}(s,\chi)x^{-s}ds=\sum_{p,m}\chi(p^m)\log p\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}F(s)(p^m x)^{-s}ds$$
$$=\sum_{p,m}(-1)^{m(p-1)/2}\cdot\log p\cdot f(xp^m),$$

where m runs over the integers $\geqq 1$. Moving the line of the integration to $\mathrm{Re}(s)=-1/2$, we get for $0<x<x_0$

$$S\equiv\sum_p(-1)^{(p-1)/2}\log p\cdot f(xp)=-\sum_p\log p\cdot f(xp^2)$$
$$-\sum_{p,m\geqq 3}(-1)^{m(p-1)/2}\log p\cdot f(xp^m)-\sum_\rho F(\rho)x^{-\rho}$$
$$+\left(-\log x\frac{L'}{L}(0,\chi)+2\Gamma'(1)\frac{L'}{L}(0,\chi)+\left(\frac{L'}{L}\right)'(0,\chi)\right)+0(x^{1/2})$$
$$=S_1+S_2+S_3+S_4+O(x^{1/2}),$$

say, where ρ runs over the non-trivial zeros of $L(s,\chi)$. We put $T=1/x$.

$$S_2\ll\sum_{p^m\leqq T,m\geqq 3}\log p\left(-\log\frac{p^m}{T}+0(1)\right)+\sum_{T<p^m\leqq T^{6/5},m\geqq 3}\log p\cdot e^{-2\sqrt{p^m/T}}(p^m/T)^{-1/4}$$
$$+\sum_{p^m\geqq T^{6/5},m\geqq 3}\log p\cdot e^{-2\sqrt{p^m/T}}(p^m/T)^{-1/4}$$
$$\ll\log^2 T\sum_{p\leqq T^{2/5}}\cdot 1+\sum_{n=2}^{\infty}\log n\cdot e^{-2n^{1/12}}$$
$$\ll T^{2/5}\log T.$$

$$S_4\ll\log T.$$

$$-S_1=\Big(\sum_{p\leq\sqrt{T}}+\sum_{p>\sqrt{T}}\Big)\log p\cdot f(xp^2)=S_5+S_6,\text{ say.}$$

$$S_5=\int_1^{\sqrt{T}}f(v^2/T)d(v+R(v))=\int_1^{\sqrt{T}}f(v^2/T)dv-2\int_1^{\sqrt{T}}f'(v^2/T)\frac{v}{T}R(v)dv$$
$$+0(\sqrt{T}\exp(-C\sqrt{\log T})),$$

where we put $\sum_{2<p<v}\log p=v+R(v)$ and C is some positive absolute constant. The last integral is

$$=\int_1^{\sqrt{T}}(-1+0(v/T))R(v)/v\,dv\ll\sqrt{T}\exp(-C\sqrt{\log T}).$$

We remark that

$$\frac{1}{\sqrt{T}}\int_1^{\sqrt{T}}f(v^2/T)dv=\int_{1/\sqrt{T}}^1\left(-2\log u-2C_0-2\log u\sum_{k=1}^{\infty}\frac{u^{2k}}{(k!)^2}\right.$$
$$\left.+2\sum_{k=1}^{\infty}\frac{u^{2k}}{(k!)^2}\psi(k+1)\right)du$$
$$=2-2C_0+2\sum_{k=1}^{\infty}\frac{1}{(k!)^2(2k+1)}\left(\frac{1}{2k+1}+\psi(k+1)\right)$$
$$+0(\log T/\sqrt{T})>1.3$$

for $T>T_0$, where we put $\psi(x)=\Gamma'/\Gamma(x)$ and C_0 is the Euler constant.

$$S_6=\int_{\sqrt{T}}^{\infty}f(v^2/T)dv-\int_{\sqrt{T}}^{\infty}f'(v^2/T)2v\cdot R(v)/T\cdot dv.$$

The last integral is $\ll\sqrt{T}\exp(-C\sqrt{\log T})$. We now assume G.R.H. for $L(s,\chi)$ and write $\rho=\frac{1}{2}+i\gamma$. Then we get

$$|S_3|\leqq x^{-1/2}\sum_\rho|F(\tfrac{1}{2}+i\gamma)|=x^{-1/2}\sum_{\gamma>0}\frac{2\pi}{\cosh(\gamma\pi)}$$

$$<0.001x^{-1/2}\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}<0.0002x^{-1/2},$$

where the first γ is known to be >6 and we have used the estimate $e^{-\pi\gamma}(\frac{1}{4}+\gamma^2)<0.001$ for $\gamma>6$ and $\sum_{\gamma>0}1/(\frac{1}{4}+\gamma^2)<1/5$. Combining all the above estimate, we get as $x\to+0$

$$S<-1.2x^{-1/2}.$$

This proves that G.R.H. for $L(s,\chi)$ implies the equality in Theorem 2.

For the proof of the converse, we notice that for $\mathrm{Re}(s)>1$,

$$F(s)\sum_p\frac{\chi(p)\log p}{p^s}=\int_0^\infty x^{s-1}(\sum_p\chi(p)\cdot\log p\cdot f(xp))dx.$$

Then we have only to use Hilfsatz of p. 2 and the same argument as in the Section 3 of Landau [5-I].

§ 3. **Proof of Theorem 1.** Let α be a positive number. We denote αe^{-x^α} by $f(x)$ and $\Gamma(s/\alpha)$ by $F(s)$. Using the same notations as in the previous section, we get

$$S_1=-\frac{1}{\sqrt{x}}\int_{\sqrt{x}}^\infty f(u^2)du+0(\sqrt{T}\exp(-C\sqrt{\log T}))$$

$$=-\frac{1}{\sqrt{x}}\cdot\frac{1}{2}\Gamma(1/2\alpha)+0(\sqrt{T}\exp(-C\sqrt{\log T})).$$

$$S_2+S_4\ll\sqrt{T}\exp(-C\sqrt{\log T}).$$

For S_3, we notice that under G.R.H.

$$|\Gamma(\rho/\alpha)|=\sqrt{2\pi}(|\gamma|/\alpha)^{(1/2\alpha)-(1/2)}e^{-\pi|\gamma|/2\alpha}|e^{A(\gamma,\alpha)}|,$$

where

$$|A(\gamma,\alpha)|\leqq\frac{\alpha}{8|\rho|}\int_0^\infty\frac{dx}{(x^2+(x/|\rho|)+1)}\leqq\frac{\alpha\pi}{2\cdot8\sqrt{\frac{1}{4}+\gamma^2}}\leqq\frac{\alpha\pi}{8\sqrt{145}}.$$

Suppose that $0<\alpha\leqq5.9$. Then $(\frac{1}{4}+\gamma^2)\gamma^{(1/2\alpha)-(1/2)}e^{-\pi\gamma/2\alpha}$ is strictly decreasing for $\gamma\geqq6$ and get

$$\left|\frac{1}{\alpha}S_3\right|\leqq\frac{A(\alpha)}{\sqrt{x}}\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}<\frac{A(\alpha)}{5\sqrt{x}},$$

where we put $A(\alpha)=145\cdot\sqrt{\pi/2}(1/\alpha)^{(1/2)+(1/2\alpha)}e^{\alpha\pi/8\sqrt{145}}6^{(1/2\alpha)-(1/2)}e^{-3\pi/\alpha}$. On the other hand by Binet's formula we get

$$(1/2\alpha)\Gamma(1/2\alpha)\geqq(1/2\alpha)^{(1/2\alpha)+(1/2)}e^{-1/2\alpha}\sqrt{2\pi}>A(\alpha)/5$$

provided that $0<\alpha\leqq4.19$. Thus for $0<\alpha\leqq4$, we get

$$\frac{1}{\alpha}S<-\frac{C}{\sqrt{x}}.$$

By p. 147 of [3] or p. 215 of [5-II], we get

$$S'\equiv\sum_{p>2}(-1)^{(p-1)/2}e^{-(xp)^\alpha}<-\frac{C}{\sqrt{x}\log(1/x)}.$$

This proves the half of Theorem 1. The rest is the same as the last part of the previous section.

§ 4. **Concluding remarks.** As is seen obviously, we have obtained, in fact, a theorem for a more general function which is suitable for the argument above. As consequences, we may replace $\xi(xp, 2)$ in Theorem 2 by $4K_0^2(2\sqrt{xp})$ or $e^{-xp/2}K_0(xp/2)$. Knapowski-Turan's function $\exp(-\log^2(xp))$ belongs to the same category. We remark only that the Mellin transform of $4K_0^2(2\sqrt{x})$ (or $e^{-x/2}K_0(x/2)$ or $\exp(-\log^2 x)$) is $\Gamma^4(s)/\Gamma(2s)$ (or $\Gamma^2(s)\sqrt{\pi}/\Gamma(s+\frac{1}{2})$ or $2e^{s^2}$, respectively). We remark finally that the condition on α in Theorem 1 may be relaxed a little if we get a numerical data of a few zeros of $L(s, \chi)$. We have in fact used only the fact that the first γ is >6.

References

[1] H. J. Bentz and J. Pintz: Quadratic residues and the distribution of prime numbers. Monatsh. Math., **90**, 91–100 (1980).

[2] P. L. Chebyshev: Oeuvres de P. L. Tchebychef I. Chelsea Publ. (1962).

[3] G. H. Hardy and J. E. Littlewood: Contribution to the theory of the Riemann zeta function and the theory of the distribution of primes. Acta Math., **41**, 119–196 (1917).

[4] S. Knapowski and P. Turan: Ueber einige Fragen der vergleichenden Primzahltheorie. Abh. aus Zahlentheorie und Analysis. Zur Erinnerung an E. Landau (1877–1938), 159–171 (1968).

[5] E. Landau: Über einige altere Vermutungen und Behauptungen in der Primzahltheorie. I, II. Math. Z., **1**, 1–24, 213–219 (1918).

[6] G. Voronoi: Sur une fonction transcendante et ses applications à la sommation de quelques séries. Ann. Sci. École Norm. Sup., **21** 207–267, 459–533 (1904).