67. *Cyclotomic Invariants for Links*[†),††)]

By Tsuyoshi KOBAYASHI,[*)] Hitoshi MURAKAMI,[**)] and Jun MURAKAMI[*)]

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1988)

In this note we construct numerical link invariants (*cyclotomic invariants*) by using solutions to the star-triangle relation for an N-state IRF model on a two-dimensional square lattice ($N=1,2,\cdots$) [3, 6]. Moreover we will show that these invariants can be defined by using Goeritz matrices and Seifert matrices. We also describe some of their properties; especially relations to the Jones polynomial [5], the Q-polynomial [1, 4], and the Kauffman polynomial [7].

Let $w(a,b,c,d;u)$ be the cyclotomic solution described in [6]. We consider a dual graph of an (unoriented) link diagram on a 2-sphere S^2. It decomposes S^2 into some regions and every region can be regarded as a tetragon. So we can assign to each region (or face) the Boltzmann weight $w(a,b,c,d;u)$ for every state on the graph as in Fig, 1. Here a state is an assignment of elements in Z/NZ to vertices in the graph.

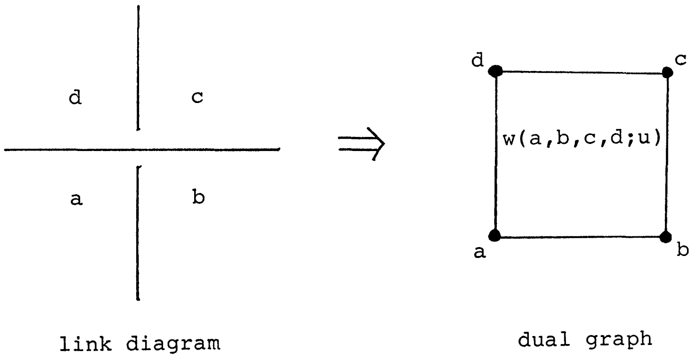

Fig. 1

This is well-defined since $w(a,b,c,d;u)=w(c,d,a,b;u)$ [6]. If we take the limit $u\to\infty\times\sqrt{-1}$ of $w(a,b,c,d;u)$, the partition function $Z_N=\sum\prod w(a,b,c,d;u)$ is invariant under the Reidemeister moves $\Omega_3^{\pm1}$ of the link diagram, where the product is taken over all the vertices of the dual graph and the sum is taken over all the states. This follows from the star-triangle relation. See [6, Fig. 2]. See also [2] for the Reidemeister moves.

---

†) Partially supported by Grant-in-Aid for Scientific Research (No. 62790098), Ministry of Education, Science and Culture, Japan.

††) Dedicated to Professor Fujitsugu HOSOKAWA on his 60th birthday.

*) Department of Mathematics, Osaka University.

**) Department of Mathematics, Osaka City University.

Now we can define the partition function directly from the link diagram as follows. Let D be a diagram on S^2 of a link in S^3. Color the regions of D with colors α and β like a chess-board. Let $\alpha_0, \alpha_1, \cdots, \alpha_n$ be the α-regions and $\beta_0, \beta_1, \cdots, \beta_m$ the β-regions. Then we can define an extended Goeritz matrix $\bar{G}=(g_{ij})$ $(0 \leqq i, j \leqq n)$ with entries g_{ij} described in [2, p. 230]. Note that $G=(g_{ij})$ $(1 \leqq i, j \leqq n)$ is a Goeritz matrix if we take $\infty \in \boldsymbol{R}^2 \cup \{\infty\}=S^2$ in α_0. Next we interchange the colors (now $\beta_0, \beta_1, \cdots, \beta_m$ are the α-regions) and define another extended Goeritz matrix $\bar{G}'=(g'_{ij})$ $(0 \leqq i, j \leqq m)$. ($G'=(g'_{ij})$ $(1 \leqq i, j \leqq m)$ is also a Goeritz matrix.) The partition function $Z_N(D)$ corresponding to the dual graph of D is now defined as follows.

$$Z_N(D)=\sum_{X,X'} \{(-1)^{\bar{g}X^T} q^{X\bar{G}X^T}\} \times \{(-1)^{\bar{g}'X'^T} q^{X'\bar{G}'X'^T}\},$$

where $q=\exp(\pi\sqrt{-1}/N)$, $\bar{g}=(g_{00}, g_{11}, \cdots, g_{nn})$, $\bar{g}'=(g'_{00}, g'_{11}, \cdots, g'_{mm})$, X (resp. X') ranges over all $1 \times (n+1)$ (resp. $1 \times (m+1)$) matrices with entries in $\boldsymbol{Z}/N\boldsymbol{Z}$, and X^T and X'^T are the transposed matrices.

Since $P\bar{G}P^T=\begin{pmatrix} 0 & O \\ O & G \end{pmatrix}$ and $P'\bar{G}'P'^T=\begin{pmatrix} 0 & O \\ O & G' \end{pmatrix}$ for some unimodular matrices of integers P and P', we have

$$Z_N(D)=N^2 \times \sum_{Y,Y'} \{(-1)^{gY^T} q^{YGY^T}\} \times \{(-1)^{g'Y'^T} q^{Y'G'Y'^T}\},$$

where $g=(g_{11}, g_{22}, \cdots, g_{nn})$, $g'=(g'_{11}, g'_{22}, \cdots, g'_{mm})$, and Y (resp. Y') ranges over all $1 \times n$ (resp. $1 \times m$) matrices with entries in $\boldsymbol{Z}/N\boldsymbol{Z}$. Now we consider an oriented link and its diagram D. Put

$$\tilde{T}_N(D)=N^{-2}(\delta_N/\sqrt{N})^{-w(D)} \times (\sqrt{N})^{-c(D)} \times Z_N(D),$$

where $\delta_N=\sum_{k=0}^{N-1}(-1)^k q^{k^2}$, $w(D)$ is the writhe of D (i.e. the algebraic sum of the crossings with $\diagup\!\!\!\!\diagdown$ being $+1$ and $\diagdown\!\!\!\!\diagup$ -1), and $c(D)$ is the number of the crossings in D. Then we have

Theorem 1. *For every integer N greater than one, $\tilde{T}_N(D)$ is an oriented link type invariant; i.e. if D and D' are diagrams of the same oriented link, then $\tilde{T}_N(D)=\tilde{T}_N(D')$.*

Proof. The invariance under the Reidemeister moves $\Omega_3^{\pm 1}$ follows from [6]. Since the invariance under $\Omega_1^{\pm 1}$ and $\Omega_2^{\pm 1}$ follows from direct computations, we omit it. Note that we can also prove the invariance under $\Omega_3^{\pm 1}$ using Goeritz matrices.

From now on we use the notation $\tilde{T}_N(L)$ instead of $\tilde{T}_N(D)$ for an oriented link L which is represented by D.

Next we use L. Traldi's modified Goeritz matrix [12] to define a "square root" of $\tilde{T}_N$. Let $H=(h_{ij})$ $(1 \leqq i, j \leqq d)$ be a modified Goeritz matrix of an oriented link L [12]. Put

$$T_N(L)=(\sqrt{N})^{-d} \sum_X \{(-1)^{hX^T} q^{XHX^T}\},$$

where $h=(h_{11}, h_{22}, \cdots, h_{dd})$ and X ranges over all $1 \times d$ matrices with entries in $\boldsymbol{Z}/N\boldsymbol{Z}$. This is well-defined (that is, independent on the choice of diagram) from [12, Theorem 1]. We call $\tilde{T}_N(L)$ and $T_N(L)$ the *cyclotomic invariants* for L. T_N is a square root of $\tilde{T}_N$ since the following holds.

Theorem 2. $\{T_N(L)\}^2=\tilde{T}_N(L)$.

Proof. We can define two modified Goeritz matrices from a diagram of L considering two types of colorings. Then the theorem follows from the definition of the modification of the Goeritz matrix in [12]. Details are omitted.

Since a modified Goeritz matrix H equals $W+W^T$ for some Seifert matrix W of L defined by using a connected Seifert surface [12] (see for example [2] for the definition of a Seifert matrix), we have

Proposition 1. *Let $W=(w_{ij})$ $(1\leqq i,\ j\leqq d)$ be a Seifert matrix of L, then*

$$T_N(L)=(\sqrt{N})^{-d}\sum_X q^{X(W+W^T)X^T},$$

where X ranges over all $1\times d$ matrices with entries in Z/NZ.

From Proposition 1, we obtain the following theorem.

Theorem 3. *Let $V_L(t)$ be the Jones polynomial of L* [5]. *Then we have*

(1) $T_2(L)=V_L(\sqrt{-1})$,

(2) $T_3(L)=V_L(\exp(\pi\sqrt{-1}/3))$, *and*

(3) $|T_p(L)|=(\sqrt{p})^{\beta_p(L)}$

for an odd prime integer p, where $\beta_p(L)$ is the first Betti number of the double branched cover of L with coefficient in Z/pZ. We can also determine the argument of $T_p(L)$ using invariants of a quadratic form (*cf.* [10]).

Proof. (1) and (2) follow from the recursive definition of $V_L(t)$ [5]. To prove (3), we remark that we may change $W+W^T$ into $P(W+W^T)P^T$ for any unimodular matrix of integers P and the entry $w_{ij}+w_{ji}$ in $W+W^T$ by N (resp. $2N$) if $i\neq j$ (resp. $i=j$) when we define $T_N(L)$ as in Proposition 1. So we can diagonalize $W+W^T$ and the conclusion follows since it is a presentation matrix for the first homology group of the double branched cover of L.

Let $F_L(a,x)$ be the Kauffman polynomial [7] and $Q_L(x)$ the Q-polynomial [1, 4]. Then from [1, 8, 9, 11] and Theorem 2 we have

Corollary.

(1) $$\tilde{T}_2(L)=\begin{cases}2^{\sharp(L)-1} & (\textit{if } L \textit{ is proper})\\ 0 & (\textit{otherwise})\end{cases}$$
$$=F_L(\exp(\pi\sqrt{-1}/4),\ -\sqrt{2})\times(-1)^{\sharp(L)-1},$$

where $\sharp(L)$ is the number of components in L and L is proper if the linking number of K and $L-K$ is even for every component K in L.

(2) $\tilde{T}_3(L)=Q_L(-1)\times(-1)^{\sharp(L)-1}=(-3)^{\beta_3(L)}\times(-1)^{\sharp(L)-1}$.

For the interpretation for $V_L(\sqrt{-1})$ see [11] and for $V_L(\exp(\pi\sqrt{-1}/3))$ see [8, 10].

Finally we remark that the cyclotomic invariants are essentially invariants for quadratic forms. So we can define them for more general situations; for example, links in homology spheres and higher dimensional links. We also remark that we may take q to be any primitive $2N$-th root of 1, which is suggested by T. Kohno.

Acknowledgement. The authors wish to thank Professor Y. Nakanishi for his helpful advice.

References

[1] R. D. Brandt, W. B. R. Lickorish, and K. C. Millett: A polynomial invariant for unoriented knots and links. Invent. Math., **84**, 563–573 (1986).

[2] G. Burde and H. Zieschang: Knots. Walter de Gruyter, Berlin and New York (1985).

[3] V. A. Fateev and A. B. Zamolodchikov: Self-dual solutions of the star-triangle relations in Z_N-models. Phys. Lett., **92**A, 37–39 (1982).

[4] C. F. Ho: A new polynomial invariant for knots and links—preliminary report. Amer. Math. Soc., Abstract 6, p. 300 (1985).

[5] V. F. R. Jones: A polynomial invariant for knots via von Neumann algebras. Bull. Amer. Math. Soc., **12**, 103–111 (1985).

[6] M. Kashiwara and T. Miwa: A class of elliptic solutions to the star-triangle relation. Nucl. Phys., B**275**[FS17], 121–134 (1986).

[7] L. H. Kauffman: An invariant of regular isotopy. Topology (to appear).

[8] W. B. R. Lickorish and K. C. Millett: Some evaluations of link polynomials. Comment. Math. Helv., **61**, 349–359 (1986).

[9] ——: An evaluation of the F-polynomial of a link (preprint).

[10] A. S. Lipson: An evaluation of a link polynomial. Math. Proc. Camb. Phil. Soc., **100**, 361–364 (1986).

[11] H. Murakami: A recursive calculation of the Arf invariant of a link. J. Math. Soc. Japan, **38**, 335–338 (1986).

[12] L. Traldi: On the Goeritz matrix of a link. Math. Z., **188**, 203–213 (1985).