31. *On Certain Subclass of Close-to-convex Functions*

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Summary. The object of the present paper is to prove a property of functions belonging to the class $\mathcal{R}_n(\alpha)$ which is the subclass of close-to-convex functions of order α in the unit disk.

1. Introduction. Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathcal{N}=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z:|z|<1\}$. A function $f(z)$ belonging to the class $\mathcal{A}_n$ is said to be convex in the unit disk $\mathcal{U}$ if and only if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z \in \mathcal{U}). \tag{1.2}$$

Further, a function $f(z)$ in the class $\mathcal{A}_n$ is said to be close-to-convex of order α ($0 \leqq \alpha < 1$) in the unit disk $\mathcal{U}$ if there exists a convex function $g(z) \in \mathcal{A}_n$ such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>\alpha \tag{1.3}$$

for some α ($0 \leqq \alpha < 1$) and for all $z \in \mathcal{U}$.

The concept of close-to-convex functions was introduced by Kaplan [2].

A function $f(z)$ belonging to $\mathcal{A}_n$ is said to be in the class $\mathcal{R}_n(\alpha)$ if and only if it satisfies

$$|f'(z)-1|<1-\alpha \tag{1.4}$$

for some α ($0 \leqq \alpha < 1$) and for all $z \in \mathcal{U}$. Noting that

$$f(z) \in \mathcal{R}_n(\alpha) \Longrightarrow \mathrm{Re}\{f'(z)\}>\alpha \qquad (z \in \mathcal{U})$$

and $g(z)=z$ is convex in $\mathcal{U}$, we see that $\mathcal{R}_n(\alpha)$ is the subclass of close-to-convex functions of order α in the unit disk $\mathcal{U}$.

Recently, Nunokawa, Fukui, Owa, Saitoh and Sekine [7] have determined the starlikeness bound of functions $f(z)$ in the class $\mathcal{R}_1(\alpha)$.

Let the functions $f(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in the unit disk $\mathcal{U}$, with $w(0)=0$ and $|w(z)|<1$ ($z \in \mathcal{U}$), such that

$$f(z)=g(w(z)) \tag{1.5}$$

for $z \in \mathcal{U}$. We denote this subordination by

$$f(z) \prec g(z). \tag{1.6}$$

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In particular, if $g(z)$ is univalent in $\mathcal{U}$, the subordination (1.6) is equivalent to $f(0)=g(0)$ and $f(\mathcal{U})\subset g(\mathcal{U})$.

This concept of subordination can be traced to Lindelöf [3], but Littlewood ([4], [5]) and Rogosinski ([8], [9]) introduced the term and discovered the basic properties.

2. Main results. In order to derive our main result, we have to recall here the following lemma due to Miller and Mocanu [6] (also Jack [1]).

Lemma. *Let the function*

$$w(z)=b_n z^n+b_{n+1}z^{n+1}+\cdots \qquad (n\in\mathcal{N}) \tag{2.1}$$

be regular in $\mathcal{U}$ *with* $w(z)\not\equiv 0$. *If* $z_0=r_0 e^{i\theta_0}$ $(r_0<1)$ *and*

$$|w(z_0)|=\max_{|z|\leqq r_0}|w(z)|,$$

then

$$z_0 w'(z_0)=m w(z_0),$$

where m *is real and* $m\geqq n\geqq 1$.

With the aid of the above lemma, we prove

Theorem. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{R}_n(\alpha)$. *Then*

$$\frac{f(z)}{z}\prec 1+\frac{(1-\alpha)z}{n+1}. \tag{2.2}$$

Proof. It is clear that the result is true if $f(z)\equiv z$. Then, we assume that $f(z)\not\equiv z$. Define the analytic function $w(z)$ in the unit disk $\mathcal{U}$ by

$$\frac{f(z)}{z}=1+\frac{(1-\alpha)w(z)}{n+1}, \tag{2.3}$$

then we see that

$$w(z)=b_n z^n+b_{n+1}z^{n+1}+\cdots$$

and $w(z)\not\equiv 0$. Now, we need only to prove that $|w(z)|<1$ for all $z\in\mathcal{U}$. If not so, there exists a point $z_0\in\mathcal{U}$ satisfying the condition of lemma such that $|w(z_0)|=1$. Therefore, applying our lemma, we have

$$z_0 w'(z_0)=m w(z_0),$$

where m is real and $m\geqq n\geqq 1$. Since, from (2.3),

$$f'(z)=1+\frac{(1-\alpha)\{zw'(z)+w(z)\}}{n+1} \tag{2.4}$$

we see that

$$\begin{aligned} f'(z_0)-1&=\frac{(1-\alpha)\{z_0w'(z_0)+w(z_0)\}}{n+1} \\ &=\frac{(1-\alpha)(m+1)w(z_0)}{n+1}, \end{aligned} \tag{2.5}$$

that is, that

$$|f'(z_0)-1|=\frac{(1-\alpha)(m+1)}{n+1}\geqq 1-\alpha. \tag{2.6}$$

This contradicts that $f(z)$ belongs to the class $\mathcal{R}_n(\alpha)$. Therefore, we complete the proof of theorem.

It follows from theorem the following

Corollary. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{R}_n(\alpha)$, *then*

$$\operatorname{Re}\left\{e^{i\beta}\frac{f(z)}{z}\right\}>0, \tag{2.7}$$

where

$$|\beta|\leqq\frac{\pi}{2}-\operatorname{Sin}^{-1}\left(\frac{1-\alpha}{n+1}\right). \tag{2.8}$$

The bound of $|\beta|$ *is best possible for the function* $f(z)$ *defined by*

$$f(z)=z+\frac{(1-\alpha)z^n}{n+1}\in\mathcal{R}_n(\alpha). \tag{2.9}$$

References

[1] I. S. Jack: Functions starlike and convex of order α. J. London Math. Soc., **3**, 469–474 (1971).
[2] W. Kaplan: Close to convex schlicht functions. Mich. Math. J., **1**, 169–185 (1952).
[3] E. Lindelöf: Mémoire sur certains inégalités dans la théorie des fonctions monogénes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel. Acta Soc. Sci. Fenn., **35**, 1–35 (1909).
[4] J. E. Littlewood: On inequalities in the theory of functions. Proc. London Math. Soc., **23**, 481–519 (1925).
[5] ——: Lectures on the Theory of Functions. Oxford University Press, London (1944).
[6] S. S. Miller and P. T. Mocanu: Second order differential inequalities in the complex plane. J. Math. Anal. Appl., **65**, 289–305 (1978).
[7] M. Nunokawa, S. Fukui, S. Owa, H. Saitoh, and T. Sekine: On the starlikeness bound of univalent functions. Math. Japonicae (to appear).
[8] W. Rogosinski: On subordinate functions. Proc. Cambridge Philos. Soc., **35**, 1–26 (1939).
[9] ——: On the coefficients of subordinate functions. Proc. London Math. Soc., **48**, 48–82 (1943).