6. A New Class of Analytic Functions Associated with the Ruscheweyh Derivatives[†]

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1. Introduction and definitions. Let $\mathcal{A}(p)$ denote the class of functions of the form

(1.1) $$f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k}z^{p+k} \qquad (p\in\mathcal{N}=\{1,2,3,\cdots\})$$

which are analytic in the unit disk $\mathcal{U}=\{z:|z|<1\}$. We denote by $f*g(z)$ the Hadamard product (or convolution) of two functions $f(z)\in\mathcal{A}(p)$ and $g(z)\in\mathcal{A}(p)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

(1.2) $$g(z)=z^p+\sum_{k=1}^{\infty} b_{p+k}z^{p+k} \qquad (p\in\mathcal{N}),$$

then

(1.3) $$f*g(z)=z^p+\sum_{k=1}^{\infty} a_{p+k}b_{p+k}z^{p+k}.$$

Following Goel and Sohi [7], we put

(1.4) $$D^{n+p-1}f(z)=\frac{z^p}{(1-z)^{n+p}}*f(z) \qquad (n>-p)$$

for the $(n+p-1)$th order Ruscheweyh derivative of $f(z)\in\mathcal{A}(p)$.

A function $f(z)\in\mathcal{A}(p)$ is said to be in the class $\mathcal{K}(n,p)$ if and only if

(1.5) $$\mathrm{Re}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right)>\frac{n+p}{2(n+1)} \qquad (z\in\mathcal{U})$$

for $n\in\mathcal{N}_0=\mathcal{N}\cup\{0\}$ and $p\in\mathcal{N}$. In particular, for $p=1$, the class $\mathcal{K}(n,1)$ becomes the class $\mathcal{K}_n$ studied by Ruscheweyh [17] who, in fact, proved the basic property [17, p. 110, Theorem 1]:

(1.6) $$\mathcal{K}_{n+1}\subset\mathcal{K}_n \qquad (n\in\mathcal{N}_0).$$

We now introduce the subclass $\mathcal{A}_{n,p}(a,b)$ of $\mathcal{A}(p)$, which is defined below by using the $(n+p-1)$th order Ruscheweyh derivative of $f(z)$.

Definition. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}(p)$, and set

(1.7) $$F_{n,p}(z)=\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n+p}{2(n+1)}$$

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for $n \in \mathcal{N}_0$ and $p \in \mathcal{N}$. Then we say that $f(z)$ is in the class $\mathcal{A}_{n,p}(a, b)$ if it satisfies the inequality

(1.8) $$\mathrm{Re}\,\{(F_{n,p}(z))^a(F_{n+1,p}(z))^b\}>0 \qquad (z \in \mathcal{U})$$

for $n \in \mathcal{N}_0$ and $p \in \mathcal{N}$; here a and b are real numbers, and each of the power functions is interpreted as its principal value.

Clearly, we have [cf. Equation (1.6)]

(1.9) $$\mathcal{A}_{n,1}(1, 0)=\mathcal{K}(n, 1)\equiv\mathcal{K}_n \quad \text{and} \quad \mathcal{A}_{n,1}(0, 1)=\mathcal{K}(n+1, 1)\equiv\mathcal{K}_{n+1}.$$

Several other classes of analytic functions defined by using the nth order Ruscheweyh derivatives of $f(z)$ have been studied in the literature by, for example, Ahuja [1], Al-Amiri ([2], [3]), Bulboaca [5], Fukui and Sakaguchi [6], Goel and Sohi ([8], [9]), Owa ([13], [14], [15]) Kumar and Shukla [10], and Singh and Singh [18].

In this paper we first present an interesting property of the class $\mathcal{A}_{n,p}(a, b)$ and then state a closely related open problem. We also study a general Libera type integral operator $\mathcal{J}_{n,p}$ defined by Equation (3.1) below.

2. A property of the class $\mathcal{A}_{n,p}(a, b)$. We first state and prove an interesting property of the class $\mathcal{A}_{n,p}(a, b)$.

Theorem 1. *Let $n \in \mathcal{N}_0$, $p \in \mathcal{N}$, $0 \leqq t \leqq 1$, and let a and b be real numbers. Then*

(2.1) $$\mathcal{A}_{n,p}(a, b)\cap\mathcal{A}_{n,p}(1, 0)\subset\mathcal{A}_{n,p}((a-1)t+1, bt).$$

Proof. Following the technique used earlier by Owa [15], let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}_{n,p}(a, b)\cap\mathcal{A}_{n,p}(1, 0)$. Also define

(2.2) $$V_{n,p}(z)=(F_{n,p}(z))^a(F_{n+1,p}(z))^b,$$

where $F_{n,p}(z)$ is given by (1.7). Since $f(z) \in \mathcal{A}_{n,p}(a, b)$, we have

(2.3) $$\mathrm{Re}\,(V_{n,p}(z))>0 \qquad (z \in \mathcal{U}).$$

We note that $f(z) \in \mathcal{A}_{n,p}(1, 0)$. This implies the inequality

(2.4) $$\mathrm{Re}\,(F_{n,p}(z))>0 \qquad (z \in \mathcal{U}).$$

Making use of (2.2), we have

(2.5) $$(F_{n,p}(z))^{(a-1)t+1}(F_{n+1,p}(z))^{bt}=(F_{n,p}(z))^{1-t}(V_{n,p}(z))^t.$$

Now we define a function $G(z)$ by

(2.6) $$G(z)=(F_{n,p}(z))^{1-t}(V_{n,p}(z))^t.$$

It is clear from (2.6) that $G(0)>0$. Consequently, using (2.3) and (2.4), we prove that

(2.7) $$|\arg\,(G(z))|\leqq(1-t)\,|\arg\,(F_{n,p}(z))|+t\,|\arg\,(V_{n,p}(z))|\leqq\frac{\pi}{2}.$$

This shows that $G(z)$ maps the unit disk $\mathcal{U}$ onto a domain which is contained in the right half-plane, that is, that $\mathrm{Re}\,(G(z))>0$. Thus we complete the proof of Theorem 1.

By taking $p=1$, $a=0$, and $b=1$ in Theorem 1, and applying (1.9) and (1.10), we readily have

Corollary 1. *Let $n \in \mathcal{N}_0$ and $0 \leqq t \leqq 1$. Then*

(2.8) $$\mathcal{K}(n+1, 1)\subset\mathcal{A}_{n,1}(1-t, t).$$

We conclude this section by stating a problem which is closely related to our theorem.

Problem. For $n \in \mathcal{N}_0$, $p \in \mathcal{N}$, and $0 \leqq t \leqq 1$, can we prove that

$$\mathcal{A}_{n,p}(a, b) \subset \mathcal{A}_{n,p}((a-1)t+1, bt)? \tag{2.9}$$

Remark. In the special case when $p=1$, we know that (2.9) holds true, that is, that

$$\mathcal{A}_{n,1}(a, b) \subset \mathcal{A}_{n,1}((a-1)t+1, bt),$$

which is proved by Al-Amiri [2], and also by Kumar and Shukla [10].

3. The integral operator $\mathcal{J}_{n,p}$. For a function $f(z)$ belonging to the class $\mathcal{A}(p)$, we define the integral operator $\mathcal{J}_{n,p}$ by (see also Owa and Srivastava [16, p. 126, Equation (2.1)])

$$\mathcal{J}_{n,p}(f) = \frac{n+p}{z^n} \int_0^z t^{n-1} f(t)\,dt \qquad (n > -p\,;\ p \in \mathcal{N}). \tag{3.1}$$

The operator $\mathcal{J}_{n,p}$, when $n \in \mathcal{N}$ and $p=1$, was introduced by Bernardi [4]. In particular, the operator $\mathcal{J}_{1,1}$ was studied by Libera [11] and Livingston [12]. For the general operator $\mathcal{J}_{n,p}$ defined by (3.1), we prove

Theorem 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{A}_{n,p}(1, 0)$ for $n > -p$ and $p \in \mathcal{N}$. Then*

$$\mathcal{J}_{n,p}(f) \in \mathcal{A}_{n+1,p}(1, 0) \qquad (n > -p\,;\ p \in \mathcal{N}). \tag{3.2}$$

Proof. We note from (1.1), (3.1), and (1.4) that, for $f(z) \in \mathcal{A}(p)$,

$$\mathcal{J}_{n,p}(f) = \left(z^p + \sum_{k=1}^{\infty} \frac{(n+p)_k}{(n+p+1)_k} z^{p+k}\right) * f(z) \tag{3.3}$$

and

$$D^{n+p-1} f(z) = \left(z^p + \sum_{k=1}^{\infty} \frac{(n+p)_k}{(1)_k} z^{p+k}\right) * f(z), \tag{3.4}$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ denotes the Pochhammer symbol. By using (3.3) and (3.4), we observe that

$$D^{n+p} \mathcal{J}_{n,p}(f) = D^{n+p-1} f(z) \tag{3.5}$$

and

$$(n+p+1) D^{n+p+1} \mathcal{J}_{n,p}(f) - D^{n+p} \mathcal{J}_{n,p}(f) = (n+p) D^{n+p} f(z). \tag{3.6}$$

Consequently, we have

$$\operatorname{Re}\left\{\frac{D^{n+p+1}\mathcal{J}_{n,p}(f)}{D^{n+p}\mathcal{J}_{n,p}(f)}\right\} > \frac{n+p}{n+p+1}\left(\frac{1}{n+p} + \frac{n+p}{2(n+1)}\right). \tag{3.7}$$

Thus we only need to show that the right-hand side of (3.8) cannot be less than $(n+p+1)/\{2(n+2)\}$, that is, that

$$\Phi(n, p) \equiv (n+2)\{2(n+1) + (n+p)^2\} - (n+1)(n+p+1)^2 \geqq 0 \tag{3.8}$$

for $n > -p$ and $p \in \mathcal{N}$. Observe that

$$\Phi(n, p) \geqq \Phi(n, 1) \geqq \Phi(-1, 1) = 0.$$

This implies the aforementioned inequality which completes the proof of Theorem 2.

Finally, setting $p=1$ in Theorem 2, and applying (1.8) and (1.9), we deduce

Corollary 2. *Let the function $f(z)$ be in the class $\mathcal{K}(n, 1)$ for $n > -1$. Then*

$$\mathcal{J}_{n,1}(f) \in \mathcal{K}(n+1, 1). \tag{3.9}$$

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