99. *A Note on a Global Version of the Coleman Embedding*

By Humio ICHIMURA
Department of Mathematics, University of Tokyo

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1986)

§1. **Introduction.** Let l be an odd prime number and $(\zeta_\nu)_{\nu\geq 1}$ be a fixed system of primitive l^ν-th root of unity with $\zeta_{\nu+1}^l=\zeta_\nu$. Let Ω_l^- be the "minus part" of the maximum pro-l abelian extension Ω_l over the cyclotomic field $\boldsymbol{Q}(\mu_{l^\infty})$ unramified outside l, and set $\mathfrak{G}=\mathrm{Gal}(\Omega_l^-/\boldsymbol{Q}(\mu_{l^\infty}))$. Let $\mathfrak{U}$ be the inertia group of an extension of l in $\Omega_l^-/\boldsymbol{Q}(\mu_{l^\infty})$, and let $\mathfrak{U}'$ be the projective limit of the principal unit group of $\boldsymbol{Q}_l(\zeta_\nu)$ w.r.t. the relative norm.

R. Coleman [1] constructed an embedding (w.r.t. the system $(\zeta_\nu)_\nu$) $[\mathrm{Col}]':\mathfrak{U}'\to\boldsymbol{Z}_l[[T]]^\times$, which is a basic tool in the theory of cyclotomic fields. By class field theory, $[\mathrm{Col}]'$ induces, naturally, an embedding $[\mathrm{Col}]:\mathfrak{U}\to\boldsymbol{Z}_l[[T]]^\times$. Under the conjecture $(\mathfrak{C})$ that $L_l(m,\omega^{1-m})\neq 0$ for any odd integer $m\geq 3$, we can extend [Col] to a homomorphism $\mathfrak{G}\to\boldsymbol{Q}_l[[T]]^\times$ as follows (where ω denotes the Teichmüller character and $L_l(s,\omega^{1-m})$ denotes the l-adic L-function): Note that for $\rho\in\mathfrak{U}$,

$$[\mathrm{Col}](\rho)=\exp\left(\sum_{\substack{m\geq 3\\ \text{odd}}}\frac{\varphi_m(\rho)}{m!}X^m\right)$$

where φ_m is the Coates-Wiles homomorphism and $X=\log(1+T)$. Let χ_m be the Kummer character w.r.t. the system of the l-units

$$\varepsilon_\nu(m)=\prod_{\substack{1\leq a\leq l^\nu\\ (a,l)=1}}(\zeta_\nu^a-1)^{a^{m-1}},$$

i.e. χ_m is a homomorphism $\mathfrak{G}\to\boldsymbol{Z}_l$ such that

$$(\varepsilon_\nu(m)^{1/l^\nu})^{\rho-1}=\zeta_\nu^{\chi_m(\rho)}$$

for any $\nu\geq 1$ and $\rho\in\mathfrak{G}$. This Kummer character is considered in Soulé [8], Deligne [3] and Ihara [5]. See, also, Ichimura-Sakaguchi [4]. By Coleman, $\chi_m|\mathfrak{U}=(1-l^{m-1})L_l(m,\omega^{1-m})\varphi_m$. Therefore, under the conjecture $(\mathfrak{C})$, the homomorphism

$$\psi:\mathfrak{G}\ni\rho\mapsto f_\rho(T)=\exp\left(\sum_{\substack{m\geq 3\\ \text{odd}}}\frac{(1-l^{m-1})^{-1}L_l(m,\omega^{1-m})^{-1}\chi_m(\rho)}{m!}X^m\right)\in\boldsymbol{Q}_l[[T^\times]]$$

is a global version of [Col], i.e. $\psi|\mathfrak{U}=[\mathrm{Col}]$.

The purpose of this note is to study some properties of ψ. Clearly, $\psi^{-1}(\boldsymbol{Z}_l[[T]]^\times)\supset\mathfrak{U}\,\mathrm{Ker}\,\psi$. But since there appear $L_l(m,\omega^{1-m})^{-1}$ in the coefficient of T^m of $f_\rho(T)$, there may be some $\rho\in\mathfrak{G}$ such that $f_\rho(T)\notin\boldsymbol{Z}_l[[T]]^\times$. The main aim of this note is to show the following

Theorem (Under the conjecture $(\mathfrak{C})$). $\psi^{-1}(\boldsymbol{Z}_l[[T]]^\times)=\mathfrak{U}\,\mathrm{Ker}\,\psi$.

Further, we prove a proposition on the kernel of ψ.

Acknowledgement. The author is very grateful to Prof. Y. Ihara for suggesting him that [Col] can be extended to ψ as stated above and

for encouraging him during the preparation of this note.

§2. **Proof of Theorem.** In this section, we always assume the conjecture ($\mathfrak{C}$). For a $Z_l[\mathrm{Gal}(Q(\zeta_l)/Q)]$-module M and an integer j, $M^{(j)}=M^{(j \bmod (l-1))}$ denotes the ω^j-eigenspace of M. For an odd integer i with $1\leq i\leq l-2$, $\psi^{(i)}$ denotes the homomorphism $\psi|\mathfrak{G}^{(i)}:\mathfrak{G}^{(i)}\rightarrow Q_l[[T]]^\times$. Then the theorem is equivalent to the following

Theorem′ (Under the conjecture ($\mathfrak{C}$)). $(\psi^{(i)})^{-1}(Z_l[[T]]^\times)=\mathfrak{U}^{(i)}\,\mathrm{Ker}\,\psi^{(i)}$.

When $i=1$, we see that $\mathfrak{G}^{(1)}=\mathfrak{U}^{(1)}$ by using the Stickelberger theorem (see e.g. Washington [9], Proposition 6.16). So, in this case, Theorem′ is obvious. In the following, we always assume $i>1$.

Before the proof for $i>1$, we recall some facts on a certain power series G_ρ. For an odd integer j and $\rho\in\mathfrak{G}^{(j)}$, set

$$G_\rho^{(j)}=\exp\left(\sum_{m\equiv j}\frac{(1-l^{m-1})^{-1}\chi_m(\rho)}{m!}X^m\right).$$

This power series has been constructed in Ihara [5], and its properties are investigated by G. Anderson, Coleman and Ihara-Kaneko-Yukinari [6]. It is known that for $i>1$, $G_\rho^{(i)}\in Z_l[[T]]^\times$ and $\mathfrak{N}(G_\rho^{(i)})=G_\rho^{(i)}$ where $\mathfrak{N}$ denotes the Coleman norm operator. So, $G_\rho^{(i)}$ ($i>1$) is a Coleman power series of some element of $\mathfrak{U}^{(i)}$.

To prove Theorem′ for $i>1$, it suffices to show that $(\psi^{(i)})^{-1}(Z_l[[T]]^\times)\subset\mathfrak{U}^{(i)}\,\mathrm{Ker}\,\psi^{(i)}$. Assume $f_\rho\in Z_l[[T]]^\times$ with $\rho\in\mathfrak{G}^{(i)}$. We easily see that $f_\rho^{g_i}=f_{\rho g_i}=G_\rho^{(i)}$ where $g_i(\in Z_l[[T]])$ is the power series such that $g_i((1+l)^s-1)=L_l(s,\omega^{1-i})$. Let λ denote the map : $Z_l[[T]]^\times\ni f\mapsto\lambda f=(1-\varphi/l)\log f\in Z_l[[T]]$, and let $\mathfrak{S}$ denote the Coleman trace operator acting on $Z_l[[T]]$. Then since $G_\rho^{(i)}$ is a Coleman power series, $0=\mathfrak{S}(\lambda G_\rho^{(i)})=\mathfrak{S}(\lambda f_\rho)^{g_i}$. From this, it is easy to see that $D^M(\mathfrak{S}(\lambda f_\rho))=0$ for some M ($<\infty$), where $D=(1+T)d/dT$. Hence, by Coleman [2], $\mathfrak{S}(D^M(\lambda f_\rho))=0$. Set $V=\{g\in Z_l[[T]]\,;\,\mathfrak{S}(g)=0\}$. By [2], $Z_l[[T]]=V+\varphi(Z_l[[T]])$ (disjoint sum) and $D(V)=V$, $D(\varphi(Z_l[[T]]))\subset\varphi(Z_l[[T]])$. Using this fact, we easily see that $D^M(\lambda f_\rho-g)=0$ for some $g\in V^{(i)}$. But since $i>1$, g comes from a Coleman power series, i.e. $g=\lambda f_\varepsilon$ for some $\varepsilon\in\mathfrak{U}^{(i)}$. Hence, as

$$\lambda f_\rho-g=\sum_{m\equiv i}\frac{L_l(m,\omega^{1-i})^{-1}\chi_m(\rho\cdot\varepsilon^{-1})}{m!}X^m$$

$\chi_m(\rho\cdot\varepsilon^{-1})=0$ except for a finite number of m's. Since χ_m is continuous in m, this implies that $\chi_m(\rho\cdot\varepsilon^{-1})=0$ for all $m\equiv i$. Therefore, $\rho\cdot\varepsilon^{-1}\in\bigcap_{m\equiv i}\mathrm{Ker}\,\chi_m=\mathrm{Ker}\,\psi^{(i)}$, hence $\rho\in\mathfrak{U}^{(i)}\,\mathrm{Ker}\,\psi^{(i)}$. This completes the proof of Theorem′.

§3. **The kernel of ψ.** Let Cyclo be the subextension of $\Omega_l^-/Q(\mu_{l^\infty})$ corresponding to $\bigcap_{m\geq 1,\ \mathrm{odd}}\mathrm{Ker}\,\chi_m$. Then under the conjecture ($\mathfrak{C}$), the field Cyclo corresponds to $\mathrm{Ker}\,\psi$. In [4] §3, we proved that Ω_l^- is unramified over Cyclo and that under the Vandiver conjecture for l, Cyclo$=\Omega_l^-$. In this section, we prove the following

Proposition. (i) Cyclo$=\Omega_l^-$ *if and only if there exist* $\rho\in\mathfrak{G}$ *and* $c\in Z_l-\{0\}$ *such that for all odd integers* $m\geq 1$, $\chi_m(\rho)=c$. (ii) *The characteristic* *power series of the torsion* Λ $(=\boldsymbol{Z}_l[[T]])$*-module* Gal $(\Omega_l^-/\text{Cyclo})$ *has no linear factors if and only if* $[\boldsymbol{Z}_l : \text{Image}\, \chi_m]$ *are bounded as odd integers* $m\to\infty$.

Remark. (1) By some computation on $\varepsilon_\nu(m)$, the Vandiver conjecture for l is valid if and only if there is $\rho\in\mathfrak{G}$ such that for all odd integers $m\geq 1$, $\chi_m(\rho)=1$. So, Proposition (i) asserts that the Vandiver conjecture for l and $\Omega_l^-=\text{Cyclo}$ are "almost" equivalent.

(2) By Soulé, $\chi_m\neq 0$, hence $[\boldsymbol{Z}_l : \text{Image}\, \chi_m]<\infty$. See [4] § 2.

Proof of the proposition. It suffices to prove the $\Delta=\text{Gal}\,(\boldsymbol{Q}(\zeta_1)/\boldsymbol{Q})$-decomposed version of the proposition. Let i be an odd integer with $1\leq i\leq l-2$. Let $\Omega_l^{(i)}$ denote the subextension of $\Omega_l^-/\boldsymbol{Q}(\mu_{l^\infty})$ corresponding to $\oplus_{j\neq i}\mathfrak{G}^{(j)}$ and set $C^{(i)}=\Omega_l^{(i)}\cap\text{Cyclo}$. For $i=1$, $C^{(1)}=\Omega_l^{(1)}$ because $\mathfrak{G}^{(1)}=\mathfrak{U}^{(1)}$ and Ω_l^-/Cyclo is unramified. Further, by some computation on $\varepsilon_\nu(m)$, we see that there exists $\rho\in\mathfrak{G}^{(1)}$ such that for all $m\equiv 1$, $\chi_m(\rho)=1$. Hence, when $i=1$, the proposition is valid. So, in the following, we assume $i>1$.

First, we assume the conjecture ($\mathfrak{C}$). Let $G^{(i)}$ denote the map: $\mathfrak{G}^{(i)}\ni\rho\mapsto G_\rho^{(i)}\in\boldsymbol{Z}_l[[T]]^\times$. Then Image $\lambda\circ G^{(i)}\subset V^{(i)}$. We easily see that torsion Λ-modules $V^{(i)}/\text{Image}\,\lambda\circ G^{(i)}$ and Ker $G^{(i)}$ $(=\text{Ker}\,\psi^{(i)})$ have the same characteristic power series by using the facts (1) Ω_l^-/Cyclo is unramified, (2) $\lambda\circ[\text{Col}](\mathfrak{U}^{(i)})=V^{(i)}$ (see [2]) and (3) $f_{\rho g_i}=G_\rho^{(i)}$. Now our assertions follow immediately from this and the facts (4) $V^{(i)}$ is a free Λ-module generated by $\sum_{m\equiv i}(1/m!)X^m$ (see [2]), (5) $\mathfrak{G}^{(i)}$ has no nontrivial finite Λ-submodule (see Iwasawa [7]).

The proof of the general case goes through similarly by considering power series

$$\exp\left(\sum_m{}'\,\frac{(1-l^{m-1})^{-1}L_l(m,\omega^{1-m})^{-1}\chi_m(\rho)}{m!}X^m\right)$$

$$\exp\left(\sum_m{}'\,\frac{\varphi_m(\rho)}{m!}X^m\right)\quad\text{and}\quad\exp\left(\sum_m{}'\,\frac{(1-l^{m-1})^{-1}\chi_m(\rho)}{m!}X^m\right)$$

instead of f_ρ, $[\text{Col}](\rho)$ and $G_\rho^{(i)}$ respectively where the sum $\sum_m'$ is taken over all natural numbers with $m\equiv i$ (mod $l-1$) and $L_l(m,\omega^{1-m})\neq 0$.

References

[1] R. Coleman: Division values in local fields. Inv. Math., **53**, 91–116 (1979).
[2] ——: Local units modulo circular units. Proc. Amer. Math. Soc., **89**, 1–7 (1983).
[3] P. Deligne: (letters to Bloch).
[4] H. Ichimura and K. Sakaguchi: The non-vanishing of certain Kummer character χ_m (after Soulé), and some related topics (to appear in Adv. St. in Pure Math., **12**).
[5] Y. Ihara: Profinite braid groups, Galois representations and complex multiplications. Ann. of Math., **123**, 43–109 (1986).
[6] Y. Ihara, M. Kaneko and A. Yukinari: On some properties of the universal power series for Jacobi sums (to appear in Adv. St. in Pure Math., **12**).
[7] K. Iwasawa: On Z_l-extensions of algebraic number fields. Ann. of Math., **98**, 246–326 (1973).
[8] C. Soulé: (letter to Ihara).
[9] L. Washington: Introduction to Cyclotomic Fields. Graduate Texts in Math., Springer-Verlag, New York (1982).