82. On Kakeya's Maximal Function

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Let $\mathcal{R}$ be a family of non-empty bounded open sets in the d-dimensional Euclidean space $\boldsymbol{R}^d$. For a locally integrable function f on $\boldsymbol{R}^d$ the maximal operator $M_{\mathcal{R}}$ with respect to $\mathcal{R}$ is defined by

$$M_{\mathcal{R}}f(x)=\sup_{x\in R\in\mathcal{R}}\frac{1}{|R|}\int_R |f|\,dy.$$

The maximal operators of this description are used effectively to estimate some operators arising, especially, in harmonic analysis. When $\mathcal{R}$ is the family of all open balls in $\boldsymbol{R}^d$, $M_{\mathcal{R}}f$ is Hardy-Littlewood maximal function. For given real numbers $N>2$ and $a>0$ let $\mathcal{R}$ be the family of rectangles in $\boldsymbol{R}^d$ with dimension $a\times\cdots\times a\times aN$, but with arbitrary direction. When $d=2$, by Cordoba's theorem (cf., e.g., [1])

$$\|M_{\mathcal{R}}f\|_2\leq C(\log N)^{1/2}\|f\|_2$$

for $f\in L^2(\boldsymbol{R}^2)$, where C is a constant independent of a, N and f, and $\|f\|_2 =\left(\int_{\boldsymbol{R}^2}|f|^2dx\right)^{1/2}$.

In this note we shall consider the higher dimensional case of Cordoba's inequality for functions of product type. We use the same notation C for a constant independent of a and N. It may be different in each occasion.

Theorem. *Let $d\geq 3$. There exists a constant C such that*

$$\|M_{\mathcal{R}}f\|_d\leq C(\log N)^{3/2}\|f\|_d$$

for all f in $L^d(\boldsymbol{R}^d)$ of the form $f(x_1,\cdots,x_d)=f_1(x_1)\cdots f_d(x_d)$.

Proof. We may assume $a=1$. Decompose $\boldsymbol{R}^d$ into cubes Q_p which have side length 1 and centers at lattice points p. We choose rectangles R_p so that each R_p has dimension $2\sqrt{d}\times\cdots\times 2\sqrt{d}\times 2N$ and center at p, and

$$M_{\mathcal{R}}f(x)\leq(2\sqrt{d})^d\sum_p\frac{1}{|R_p|}\int_{R_p}|f|\,dy\cdot\chi_{Q_p}(x), \tag{1}$$

where χ_E denotes the characteristic function of a set E. Let $Tf(x)$ be the sum on the right hand side of (1). Fix $1\leq i<j\leq d$. For $x=(x_1,\cdots,x_d)$ denote $\bar{x}=(x_i,x_j)$ and $\bar{\bar{x}}=(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_{j-1},x_{j+1},\cdots,x_d)$. We shall prove that

$$\int_{\boldsymbol{R}^2}(\sup_{\bar{\bar{x}}}Tf)^2d\bar{x}\leq C(\log N)^3\int_{\boldsymbol{R}^2}(\sup_{\bar{\bar{x}}}|f|)^2d\bar{x}. \tag{2}$$

Then by an interpolation theorem for operators on mixed normed spaces given in the previous paper [2] we get our theorem for functions f of product type.

To show (2) we consider the dual operator $T^*a(x)=\sum_p a_p|R_p|^{-1}\chi_{R_p}(x)$ for sequences $a=\{a_p\}$ of complex numbers. We shall prove that

$$\left\{\int_{R^2}\left(\int_{R^{d-2}}|T^*a|^2\,d\bar{\bar{x}}\right)^2 d\bar{x}\right\}^{1/2}\le C(\log N)^{3/2}\{\sum_{\bar{p}}(\sum_{\bar{\bar{p}}}|a_p|)^2\}^{1/2}. \tag{3}$$

Lemma 1. *There exist rectangles S_p in $\boldsymbol{R}^2$ such that*

(i) *S_p has dimension $2\sqrt{d}\times M_p$ with $2<M_p<2N$ and center at $\bar{p}$, and*

(ii)
$$\int_{R^{d-2}}|T^*a(x)|\,d\bar{\bar{x}}\le C\sum_p |a_p||S_p|^{-1}\chi_{S_p}(\bar{x}). \tag{4}$$

Proof. Fix a rectangle R_p and assume $p=0$. Let $B=\{|x|<2\sqrt{d}\}$. Then we can choose a unit vector u in $\boldsymbol{R}^d$ so that $R_p\subset B+\{ju:|j|\le N\}=E$, say. Put $\rho(\bar{x})=\int \chi_E(x)d\bar{\bar{x}}$ and $S_p=\{\bar{x}:|\bar{x}-j\bar{u}|<2\sqrt{d}$ for some $|j|\le N\}$. Then support of $\rho\subset S_p$ and $\rho(\bar{x})\le C/|\bar{u}|$. Therefore $\rho(\bar{x})\le C|R_p|/|S_p|$, from which Lemm 1 follows.

Now we introduce a function $\bar{T}^*a(\bar{x})=\sum_p a_p|S_p|^{-1}\chi_{S_p}(\bar{x})$.

Lemma 2. *Let $k\ge 1$ and assume $2^k\le|S_p|<2^{k+1}$ for all p. Then*

$$\int_{R^2}|\bar{T}^*a|^2\,d\bar{x}\le Ck\sum_p(\sum_{\bar{\bar{p}}}|a_p|)^2.$$

Proof. Put $P_0=\{\bar{p}=(p_i,p_j)\in \boldsymbol{Z}^2:|p_i|,|p_j|\le 2^k\}$ and $P_\mu=P_0+2^{k+1}\mu$ for $\mu\in\boldsymbol{Z}^2$. Since $|S_p\cap S_q|=0$ if $|\bar{p}-\bar{q}|>2^{k+1}$, we have

$$\begin{aligned}\int|\bar{T}^*a|^2\,d\bar{x}&\le 2^{-2k}\sum_\mu\int(\sum_{\bar{p}\in P_\mu}\sum_{\bar{\bar{p}}}|a_p|\chi_{S_p}(\bar{x}))^2d\bar{x}\\&\le 2^{-k+1}\sum_\mu\sum_{p_i}\int(\sum_{p_j}\sum_{\bar{\bar{p}}}|a_p|\chi_{S_p}(\bar{x}))^2d\bar{x},\end{aligned} \tag{5}$$

where $(p_i,p_j)=\bar{p}$ runs over P_μ. We may assume $|S_p\cap S_q|\le C2^k/(|p_j-q_j|+1)$ for $\bar{p}=(p_i,p_j)$, $\bar{q}=(p_i,q_j)\in P_\mu$ by a geometric consideration (cf. [1]). Thus the right hand side of (5) does not exceed

$$C\sum_\mu\sum_{p_i}\sum_{p_j,q_j}(\sum_{\bar{\bar{p}}}|a_p|)(\sum_{\bar{\bar{q}}}|a_q|)/(|p_j-q_j|+1)\le Ck\sum_\mu\sum_{\bar{p}\in P_\mu}(\sum_{\bar{\bar{p}}}|a_p|)^2,$$

which proves Lemma 2.

Put $\bar{T}_k^*a=\sum^k a_p|S_p|^{-1}\chi_{S_p}$, where $\sum^k$ denotes the summation over p such that $2^k\le|S_p|<2^{k+1}$. Then, by Lemma 2, the left hand side of (3) is dominated by

$$\sum_{k=1}^{C+\log N}\left(\int|\bar{T}_k^*a|^2\,d\bar{x}\right)^{1/2}\le C\sum_{k=1}^{C+\log N}\sqrt{k}\{(\sum_{\bar{p}}\sum_{\bar{\bar{p}}}|a_p|)^2\}^{1/2},$$

which implies (3).

References

[1] A. Cordoba: The Kakeya maximal function and the spherical summation multipliers. Amer. J. Math., **99**, 1–22 (1977).

[2] S. Igari: Interpolation of linear operators in Lebesgue spaces with mixed norm. Proc. Japan Acad., **62**A, 46–48 (1986) (A detailed proof will appear in Tôhoku Math. J., **38**, 469–490 (1986)).