88. A Note on the Mean Value of the Zeta and L-functions. IV

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University, Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., Oct. 13, 1986)

1. In the previous notes of this series we studied the possibility of extending Atkinson's method [1] to Dirichlet L-functions. Here we turn to the more basical problem of strengthening Atkinson's result itself.

First we introduce some notations: Let T be a large parameter, and put

$$E(T)=\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt - T\left(\frac{T}{2\pi}+2\gamma-1\right),$$

where γ is the Euler constant. Let δ be a small positive constant, and $\delta\leqq\alpha<\beta\leqq 1-\delta$. We define $\lambda(x)$ on the unit interval of x to be equal to 1 if $x\leqq\alpha$, to $(\beta-\alpha)(\beta-x)$ if $\alpha\leqq x\leqq\beta$, and to 0 if $\beta\leqq x$. Then we put $\omega(n)=\lambda(2\pi n/T)$ and $\bar{\omega}(n)=1-\lambda(\exp(-2\sinh^{-1}((\pi n/2T)^{1/2})))$. Also we use the standard

$$\Delta(x)=\sum_{n\leqq x}{}' d(n)-x(\log x+2\gamma-1)-\frac{1}{4},$$

where d is the divisor function.

Then our main result is embodied in

Theorem. *There is an absolute constant c_0 such that*

$$\begin{aligned}E(T)=&\frac{1}{\sqrt{2}}\sum_{m\leqq T(\alpha)}(-1)^m\bar{\omega}(m)d(m)m^{-1/2}(\sinh^{-1}((\pi m/2T)^{1/2}))^{-1}\left(T/2\pi m+\frac{1}{4}\right)^{-1/4}\\ &\times\cos(2TF(\pi m/2T)-\pi/4)\\ &-2\sum_{n\leqq\beta T/2\pi}\omega(n)d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos\left(T\log\frac{T}{2\pi n}-T+\frac{\pi}{4}\right)\\ &+c_0+O(T^{-1/4})+O((\beta-\alpha)^{-1}(1+(\beta-\alpha)^{1/2}\log^{3/2}T)T^{-1/2}\log T),\end{aligned}$$

where $T(\alpha)=(2\pi\alpha)^{-1}(1-\alpha)^2T$ and $F(x)=\sinh^{-1}(x^{1/2})+(x(x+1))^{1/2}$; the O-constants may possibly depend on δ.

Corollary.

$$\int_0^T E(u)^2 du=\frac{1}{3}\left(\frac{2}{\pi}\right)^{1/2}\zeta^4\left(\frac{3}{2}\right)\zeta^{-1}(3)T^{3/2}+O(T\log^5 T).$$

Remark. Independently from us Meurman [4] has recently proved a result on $E(T)$ which is essentially the case $\beta-\alpha\approx T^{-1/4}$ in our theorem, and obtained the same result as our corollary an improvement upon Heath-Brown [2]. Meurman's argument is a natural refinement of Atkinson's, and in several respects simpler than ours. Our proof is based on our approximate functional equation for $\zeta^2(s)$ ([5] and [6]), and provides an alternative proof of Atkinson's original result, for the choice $\beta-\alpha=T^{-1/2}$ gives it with an improved error-term $O(\log T)$.

Acknowledgement. We are much indebted to Dr. T. Meurman of Turku University, Finland, for indicating an error contained in the earlier version of this note.

2. The deduction of the corollary from the theorem is easy. So we show the outline of the proof of the theorem only; the details will be given in our lectures to be delivered at Colorado University (Spring semester, '87).

Let $t\geqq 1$ and put

$$A(t)=2\,\mathrm{Re}\left\{\chi\left(\frac{1}{2}-it\right)\sum_{n\leqq t/2\pi} d(n)n^{-(1/2)-it}\right\},$$

$$B(t)=\left|\zeta\left(\frac{1}{2}+it\right)\right|^2-A(t),$$

where $\chi(s)=2^s\pi^{s-1}\Gamma(1-s)\sin(s\pi/2)$. An asymptotic evaluation of $B(t)$ with an error $O(t^{-3/2}\log t)$ can be obtained by following the analysis developed in [6], and without much difficulty we can show that

$$\int_1^T B(t)\,dt=\frac{1}{3}\left(\frac{T}{\pi}\right)^{1/2}\left(\log\frac{T}{2\pi}+2\gamma+4\right)+C_1(T)+c+O(T^{-1/4}),$$

where c is an absolute constant, and

$$C_1(T)=-\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{T}{2\pi}\right)^{1/4}\sum_{n=1}^{\infty} d(n)n^{-3/4}\cos\left(2(2\pi nT)^{1/2}+\frac{\pi}{4}\right)\int_0^\infty \frac{\cos(\xi+\pi/4)}{(\xi+n\pi)^{1/2}}\,d\xi.$$

As for $A(t)$ we have

$$\int_1^T A(t)dt=2\,\mathrm{Re}\left\{\sum_{n\leqq T/2\pi} d(n)n^{-1/2}\int_{2\pi n}^T \chi\left(\frac{1}{2}-it\right)n^{-it}dt\right\}.$$

A simple application of the saddle point method to the last integral yields

$$\int_1^T A(t)\,dt=2\pi\sum_{n\leqq T/2\pi} d(n)\left(1+\frac{\sqrt{2}}{6\pi}n^{-1/2}\right)+C_2(T)+c+O(T^{-1/2}(\log T)^2);$$

$$C_2(T)=-2T\,\mathrm{Re}\left\{\exp\left(iT\log\frac{T}{2\pi e}\right)\int_0^{T^{-2/5}}\left(1+\frac{T}{6}r^3\varepsilon\right)\right.$$
$$\left.\times\exp\left(i\varepsilon rT\log\frac{T}{2\pi}-\frac{T}{2}r^2\right)\sum_{n\leqq T/2\pi} d(n)n^{-(1/2)-i(1+\varepsilon r)T}dr\right\},$$

where $\varepsilon=\exp(\pi i/4)$. And combining these we get

$$E(T)=\left(\frac{T}{\pi}\right)^{1/2}\left(\log\frac{T}{2\pi}+2\gamma\right)+2\pi\left(1+\frac{1}{3}(\pi T)^{-1/2}\right)\Delta\left(\frac{T}{2\pi}\right)+\pi d\left(\frac{T}{2\pi}\right)$$
$$+C_1(T)+C_2(T)+c+O(T^{-1/4}),$$

where $d(x)$ is defined to be zero when x is not an integer. Thus, we have to evaluate $C_2(T)$ asymptotically. For this sake we divide it into two parts $C_2^{(1)}(T,\xi)$ and $C_2^{(2)}(T,\xi)$ according to $n\leqq\xi T/2\pi$ and $\xi T/2\pi<n\leqq T/2\pi$, respectively, with $\alpha\leqq\xi\leqq\beta$. Then we have the trivial

$$C_2(T)=(\beta-\alpha)^{-1}\int_\alpha^\beta C_2^{(1)}(T,\xi)\,d\xi+(\beta-\alpha)^{-1}\int_\alpha^\beta C_2^{(2)}(T,\xi)d\xi$$
$$=C_2^{(1)}(T)+C_2^{(2)}(T),$$

say; this perturbation device induces a lot of cancellations. It is easy to see that $C_2^{(1)}(T)$ is essentially the sum over n in our theorem. To $C_2^{(2)}(T)$ we apply the device of introducing the trivial factor $\exp(2\pi ni)$, due to Jutila [3], and by partial summation we get, with the aid of Tong's result on the mean square of $\Delta(x)$,

$$C_2^{(2)}(T) = -\left(\frac{T}{\pi}\right)^{1/2}\left(\log\frac{T}{2\pi}+2\gamma\right)-2\pi\left(1+\frac{1}{3}(\pi T)^{-1/2}\right)\Delta\left(\frac{T}{2\pi}\right)-\pi d\left(\frac{T}{2\pi}\right)$$
$$+C_2^{(3)}(T)+O(T^{-1/4})+O((\beta-\alpha)^{-1/2}T^{-1/2}\log^{5/2}T),$$

where

$$C_2^{(3)}(T)=2T\,\mathrm{Re}\left\{\exp\left(iT\log\frac{T}{2\pi e}\right)\int_0^{T^{-2/5}}\left(1+\frac{T}{6}\varepsilon r^3\right)\exp\left(i\varepsilon\left(T\log\frac{T}{2\pi}\right)r-\frac{T}{2}r^2\right)\right.$$
$$\left.\times\int_{\alpha T/2\pi}^{T/2\pi}(1-\omega(x))\Delta(x)d(x^{-1/2}\exp(2\pi ix-iT(1+\varepsilon r)\log x))dr\right\}.$$

Then, by Voronoi's formula for $\Delta(x)$, the problem is reduced, essentially, to the computation of the integral

$$\int_{\alpha T/2\pi}^{T/2\pi}(1-\omega(x))x^{1/4}\exp(\pm 4\pi i(nx)^{1/2})d(x^{-1/2}\exp(2\pi ix-iT(1+\varepsilon r)\log x)),$$

where $n\geqq 1$ is an integer. To this we apply the saddle point method. The saddle point is at

$$x=n/2+T(1+\sqrt{2}\,r)/2\pi-(n^2/4+Tn(1+\sqrt{2}\,r)/2\pi)^{1/2}.$$

Thus the relevant range of n is $Tr^2/\pi\leqq n\leqq T(2\pi\alpha)^{-1}(1-\alpha+\sqrt{2}\,r)^2$. Estimation of the resulting integrals around the points $x=\alpha T/2\pi$ and $\beta T/2\pi$ as well as those along line segments well off the real axis is of no problem. But the estimation of the integrals around the saddle point and $T/2\pi$ is quite involved, especially when n is close to Tr^2/π. Nevertheless, we can show that

$$C_2(T)=(\text{the first term of the formula in the theorem})-C_1(T)$$
$$+O(T^{-1/4})+O((\beta-\alpha)^{-1}T^{-1/2}\log T),$$

which ends the proof of the theorem.

References

[1] F. V. Atkinson: The mean value of the Riemann zeta-function. Acta Math., **81**, 353–376 (1949).

[2] D. R. Heath-Brown: The mean value theorem for the Riemann zeta-function. Mathematika, **25**, 177–184 (1978).

[3] M. Jutila: Transformation formulae for Dirichlet polynomials. J. Number Theory, **18**, 135–156 (1984).

[4] T. Meurman: On the mean square of the Riemann zeta-function (to appear).

[5] Y. Motohashi: A note on the approximate functional equation for $\zeta^2(s)$. II. Proc. Japan Acad., **59A**, 469–472 (1983).

[6] ——: An asymptotic expansion for $\zeta^2(s)$ (manuscript).