78. On the Number of Prime Factors of Integers in Short Intervals

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1. **Introduction.** Let $3\leqslant n<m$ be integers. Let $\omega(m)$ denote the number of distinct prime factors of m. Let $1<b(n)\leqslant n$ be a sequence of positive integers. Let $A\{m;\cdots\}$ denote the number of positive integers m which satisfy some conditions. Throughout this paper $p, p_1, p_2, \cdots$ stand for prime numbers and $c_1, c_2, \cdots$ stand for positive constants. We put

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-(1/2)y^2}dy.$$

Then the following result was obtained by Babu [1].

Let $1\leqslant a(n)\leqslant(\log\log n)^{1/2}$ be a sequence of real numbers tending to infinity. Then

(1) $(1/b(n))A\{m;\ n<m\leqslant n+b(n),\ \omega(m)-\log\log m<x\sqrt{\log\log m}\}\longrightarrow\Phi(x)$

as $n\to\infty$, provided that $b(n)\geqslant n^{a(n)(\log\log n)^{-1/2}}$.

In this note we shall prove the following theorem which shows that the condition for $b(n)$ can be improved.

Theorem. *Let $\alpha<\beta$ be real numbers. Let $b(n)\geqslant n^{1/(\log\log n)}$ be a sequence of positive integers. We put $\mu=\max\{1,|\alpha|,|\beta|\}$ and*

$$A(n, b(n), \alpha, \beta)=A\{m;\ n<m\leqslant n+b(n),$$
$$\log\log m+\alpha\sqrt{\log\log m}<\omega(m)<\log\log m+\beta\sqrt{\log\log m}\}.$$

Then we have

$$\frac{1}{b(n)}A(n, b(n), \alpha, \beta)=\frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{-(1/2)y^2}dy+O\left(\frac{\mu^5(\log\log\log n)^{1/2}}{(\log\log n)^{1/4}}\right)$$
$$+O(\mu\sqrt{\log\log n}\, e^{-c_1(\log\log n)^2\log b(n)/\log n}).$$

The O-terms are uniform with respect to a sufficiently large n.

This theorem implies that (1) holds if $b(n)\geqslant n^{1/\log\log n}$, and also gives an answer for the question which was given by P. Erdös and I. Z. Ruzsa (cf. [1]). To prove the theorem we shall use Selberg's sieve method and the arguments of Erdös [3] and Tanaka [5] (cf. [2]).

2. **Sieve method.** Using Kubilius's lemma (Kubilius [3], lemma 1.4) we obtain the following lemma. This also can be proved directly by Selberg's sieve method.

Lemma. *Let $b_1(n)$ be a sequence of positive integers tending to infinity. Let $g\leqslant\sqrt{b_1(n)}$ be a positive integer and q be an integer with $0\leqslant q<g$. Let $n_1=[(n-q)/g]$ and $n_2=[(n+b_1(n)-q)/g]$, here $[x]$ denotes the largest integer not exceeding x. Let $r_1\geqslant 2$ with $\log r_1\leqslant c_2\log(n_2-n_1)$, where*

c_2 is a sufficiently small constant. Let $p_1, p_2, \cdots, p_h$ be prime numbers such that $p_j \nmid g$ and $p_j \leqslant r_1$ for each $j=1, 2, \cdots, h$. We put $F(n, b_1(n), q, g; p_1, p_2, \cdots, p_h) = A\{m; n<m\leqslant n+b_1(n), m\equiv q \pmod g, m\not\equiv 0 \pmod{p_j}\ j=1, 2, \cdots, h\}$. Then we have

$$F(n, b_1(n), q, g; p_1, p_2, \cdots, p_h) = \frac{b_1(n)}{g}\prod_{j=1}^{h}\left(1-\frac{1}{p_j}\right)\{1+O(e^{-c_3(\log b_1(n)/\log r_1)})\}.$$

The O-term is uniform with respect to a sufficiently large n and $g\leqslant\sqrt{b_1(n)}$.

3. **Proof of Theorem.** We denote by $P=P(n)$ a set of all prime numbers p which satisfy an inequality

$$\log n < p < n^{1/(8(\log\log n)^2)}.$$

Let $\omega'(m)$ be the number of distinct primes in P which are divisors of m. Let $g(n)$ be a sequence of real numbers tending to infinity. Then we have

$$(2)\qquad A\{m; n<m\leqslant n+b(n), \omega(m)-\omega'(m)>g(n)\} = O\left(\frac{b(n)\log\log\log n}{g(n)}\right).$$

Let $y(n)=\sum_{p\in P} 1/p$. Then $y(n)=\log\log n+O(\log\log\log n)$. Let t be a positive integer with $t<2\log\log n$. Let $L(t)$ be a set of all positive square-free integers which have exactly t prime factors belonging to the set P. Then we have

$$(3)\qquad \sum_{l\in L(t)}\frac{1}{l}=\frac{y(n)^t}{t!}+O\left(\frac{1}{\log\log n}\right).$$

Let $N(n, b(n), t)=A\{m; n<m\leqslant n+b(n), \omega'(m)=t\}$, and $N_1(n, b(n), t) =A\{m; n<m\leqslant n+b(n), \omega'(m)=t, p^2\nmid m \text{ for all } p\in P\}$. Then we have

$$(4)\qquad N(n, b(n), t)=N_1(n, b(n), t)+O\left(\frac{b(n)}{\log\log n}\right)$$

and

$$\begin{aligned} N_1(n, b(n), t) &= \sum_{l\in L(t)} A\{m; n<m\leqslant n+b(n), l\mid m, p\nmid m/l \text{ for all } p\in P\}\\ &= \sum_{l\in L(t)}\sum_{i=1}^{\varphi(l)} F(n, b(n), q_i l, l^2; p_1, p_2, \cdots, p_h)\end{aligned}$$

where $\{q_1, q_2, \cdots, q_{\varphi(l)}\}$ is a reduced set of residues modulo l, and p_i $(1\leqslant i \leqslant h)$ are all the prime numbers such that $p_i\nmid l$ and $p_i\in P$.

If $b(n)\geqslant n^{1/\log\log n}$, then $l^2<b(n)^{1/2}$ for any $l\in L(t)$. Hence by (3), (4) and the lemma we have

$$(5)\qquad N(n, b(n), t)=b(n)e^{-y(n)}\frac{y(n)^t}{t!}+O\left(\frac{b(n)}{\log\log n}\right) + O(b(n)e^{-c_4(\log\log n)^2\log b(n)/\log n}).$$

Let u be a real number such that $t=y(n)+u\sqrt{y(n)}$. Applying Stirling's formula to (5), we have

$$(6)\qquad N(n, b(n), t)=\frac{1}{\sqrt{2\pi y(n)}}b(n)e^{-(1/2)u^2}+O\left(\frac{\mu^4 b(n)}{\log\log n}\right) + O(b(n)e^{-c_4(\log\log n)^2\log b(n)/\log n}).$$

Now we put $w=(g(n)+\mu\log\log\log n)/\sqrt{y(n)}$, provided that the function $g(n)$ has the property that w becomes sufficiently small for a large n.

Let $B(n, b(n), \alpha, \beta) = A\{m\,;\, n<m \leqslant n+b(n),\ y(n)+\alpha\sqrt{y(n)} < \omega'(m) < y(n)+\beta\sqrt{y(n)}\}$. From (2) we have

$$(7)\quad A(n, b(n), \alpha, \beta) = B(n, b(n), \alpha+O(w), \beta+O(w)) + O\left(\frac{b(n)\log\log\log n}{g(n)}\right).$$

Let $t=t_0+1, t_0+2, \cdots, t_0+s$ be s natural numbers such that

$$y(n)+\alpha\sqrt{y(n)}<t<y(n)+\beta\sqrt{y(n)}.$$

Further, we write $t_0+i=y(n)+u_i\sqrt{y(n)}$. It is obvious that

$$u_{i+1}-u_i=1/\sqrt{y(n)} \quad \text{and} \quad s=O(\mu\sqrt{\log\log n}).$$

Hence from (6) we have

$$(8)\quad \begin{aligned} B(n, b(n), \alpha, \beta) &= \sum_{i=1}^{s} N(n, b(n), t_0+i) \\ &= \frac{b(n)}{\sqrt{2\pi}} \sum_{i=1}^{s} (u_{i+1}-u_i)\, e^{-(1/2)u_i^2} + O\left(\frac{\mu^5 b(n)}{\sqrt{\log\log n}}\right) \\ &\quad + O(b(n)\mu\sqrt{\log\log n}\, e^{-c_5(\log\log n)^2 \log b(n)/\log n}). \end{aligned}$$

Using the mean value theorem in calculus, we have

$$\sum_{i=1}^{s}(u_{i+1}-u_i)e^{-(1/2)u_i^2} = \int_{\alpha}^{\beta} e^{-(1/2)u^2}du + O\left(\frac{\mu^2}{\sqrt{y(n)}}\right).$$

Therefore the proof of theorem is completed by (7), (8) and putting $g(n) = (\log\log n)^{1/4}(\log\log\log n)^{1/2}$.

References

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