43. A Formula of Eigenfunction Expansions II.

Exterior Dirichlet Problem in a Lattice

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We apply the method used in my previous note to exterior Dirichlet problems in a lattice. It is shown there is no point spectrum.

1. Let Γ be a free abelian group with g generators $\sigma_1, \cdots, \sigma_g$ and A_0 be a self-adjoint bounded linear operator on $l^2(\Gamma)$ described by a symmetric stochastic walk on Γ:

$$A_0 u(\gamma)=\sum_{i=1}^{g} p_i(u(\gamma\sigma_i)+u(\gamma\sigma_i^{-1})). \tag{1.1}$$

Let A be the restriction of A_0 on $l^2(\Gamma-\Omega)$ corresponding to the exterior Dirichlet problem outside a finite subset Ω. Physically this corresponds to a random walk with traps Ω (see [5]). The Green function for A_0 is described by the Fourier integral formula

$$G_0(\gamma,\gamma'|z)=\frac{1}{(2\pi i)^g}\int_{S^1\times\cdots\times S^1}\frac{\omega_1^{-m_1+m_1'}\cdots\omega_g^{-m_g+m_g'}}{z-\sum_1^g p_j(\omega_j+\omega_j^{-1})}\cdot\frac{d\omega_1}{\omega_1}\wedge\cdots\wedge\frac{d\omega_g}{\omega_g} \tag{1.2}$$

for $\gamma=\sigma_1^{m_1}\cdots\sigma_g^{m_g}$ and $\gamma'=\sigma_1^{m_1'}\cdots\sigma_g^{m_g'}$ where $z\in \boldsymbol{C}-[-1,1]$. The integral depends only on $|m_1-m_1'|, \cdots, |m_g-m_g'|$.

Let S^{g-1} be the unit sphere of dimension $g-1$ and $S^{g-1}(\varepsilon_1, \cdots, \varepsilon_g)$ be the quadrant of S^{g-1} consisting of points $(\xi_1, \cdots, \xi_g)\in S^{g-1}$ such that $\varepsilon_1\xi_1>0, \cdots, \varepsilon_g\xi_g>0$ for $\varepsilon_j=\pm 1$. We denote by V_z the analytic hypersurface (so called complex Fermi hypersurface) in $(\boldsymbol{C}^*)^g$ defined by

$$F(z,\omega,\omega^{-1})\equiv z-\sum_{j=1}^{g} p_j(\omega_j+\omega_j^{-1})=0. \tag{1.3}$$

For a given direction at infinity $\xi=(\xi_1, \cdots, \xi_g)\in S^{g-1}(\varepsilon_1, \cdots, \varepsilon_g)$ consider the following equation with respect to the variables $\omega_j=\exp(\sqrt{-1}\theta_j)$ which is the inverse of the Gauss map κ from V_z to S^{g-1}:

$$\frac{1}{i}\frac{\partial F}{\partial\theta_j}\left(\equiv\omega_j\frac{\partial F}{\partial\omega_j}\right)=\xi_j\rho, \qquad 1\leq j\leq g \tag{1.4}$$

for an unknown ρ. This has the following solution $\hat{\omega}=(\hat{\omega}_1, \cdots, \hat{\omega}_g)\in V_z$:

$$\hat{\omega}_j=\frac{-\varepsilon_j\xi_j\rho+\sqrt{(\rho\xi_j)^2+4p_j^2}}{2p_j} \tag{1.5}$$

where ρ denotes the unique solution of the equation

$$\sum_{j=1}^{g}\sqrt{\zeta_j^2+4p_j^2}=z \qquad \text{for } \zeta_j=\xi_j\rho \tag{1.6}$$

such that $\rho>0$ for $z>1$.

By saddle point method and Lagrangean analysis for the Hamiltonian $\mathrm{I}_m\sum_1^g m_j'\log\omega_j$ in the Kähler manifold V_z ([1]), we can prove

Proposition 1. *We assume* $z \notin [-1,1]$. *We fix* $\xi=(\xi_1,\cdots,\xi_g)\in S^{g-1}$ *such that* $\xi_1\cdots\xi_g\neq 0$. *When the ratio* $m'_1:\cdots:m'_g$ *converges to* $\xi_1:\cdots:\xi_g$ *at the infinity in the sense that for any* $i\neq j$,

$$(1.7)\qquad \lim_{|m'_1|+\cdots+|m'_g|\to\infty} m'_j/m'_i \to \xi_j/\xi_i,$$

then the Green function $G_0(\gamma,\gamma'|z)$ *has the asymptotic behaviour in the direction* ξ

$$(1.8)\qquad G_0(e,\gamma'|z)\sim\left(\frac{\rho}{2\tau\pi}\right)^{g-1}\cdot\left\{\sum_{j=1}^{g}\frac{p_j(\hat\omega_j^{-1}-\hat\omega_j)^2}{(\hat\omega_j^{-1}+\hat\omega_j)}\right\}^{-1}$$

$$\cdot\left\{\prod_{j=1}^{g}p_j(\hat\omega_j^{-1}+\hat\omega_j)\right\}^{-1}\cdot\prod_{j=1}^{g}\hat\omega_j^{m'_j},\qquad \text{for } \tau=\sqrt{m_1'^2+\cdots+m_g'^2},$$

and the basic eigenfunction $K_0(\gamma,\xi|z)$ *has the simple form*

$$(1.9)\qquad K_0(\gamma,\xi|z)=\lim_{\gamma'\to\xi}\frac{G_0(\gamma,\gamma'|z)}{G_0(e,\gamma'|z)}=\prod_{j=1}^{g}(\hat\omega_j)^{-m_j\varepsilon_j}.$$

The behaviour of $G_0(\gamma,\gamma'|z)$ along $[-1,1]$ is more or less known and follows from its monodromic property obtained from the standard technique of Picard-Lefschetz transformations and Gauss-Manin systems (sometimes called holonomic systems) (see [5]). The result is as follows.

Lemma 1. *Assume that* $\varepsilon_1 p_1+\cdots+\varepsilon_g p_g$ *are different from each other for* $\varepsilon_j=\pm 1$. *In each domain* $\mathrm{I}_m z\geq 0$ *or* $\mathrm{I}_m z\leq 0$, *the function* $G_0(\gamma,\gamma'|z)$ *is holomorphically extendable along* $[-1,1]-\bigcup\{\pm 2p_1\pm\cdots\pm 2p_g\}$ *and has the singularities at* $z=2p_1\varepsilon_1+\cdots+2p_g\varepsilon_g$, $\varepsilon_j=\pm 1$.

$$(1.10)\qquad G_0(\gamma,\gamma'|z)\sim\prod_{j=1}^{g}(-1)^{(m'_j-m_j)\varepsilon_j}\cdot C(\varepsilon_1,\cdots,\varepsilon_g)(z-2p_1\varepsilon_1-\cdots-2p_g\varepsilon_g)^{(g-2)/2}$$

$$+t'_\pm(\gamma,\gamma'),\qquad \text{for } g \text{ odd and}$$

$$(1.11)\qquad \sim\prod_{j=1}^{g}(-1)^{(m'_j-m_j)\varepsilon_j}\cdot C(\varepsilon_1,\cdots,\varepsilon_g)(z-2p_1\varepsilon_1-\cdots-2p_g\varepsilon_g)^{(g-2)/2}$$

$$\cdot\log(z-2p_1\varepsilon_1-\cdots-2p_g\varepsilon_g)+t'_\pm(\gamma,\gamma'),\qquad \text{for } g \text{ even}$$

according as $z\to 2p_1\varepsilon_1+\cdots+2p_g\varepsilon_g\pm i0$. *Here* $C(\varepsilon_1,\cdots,\varepsilon_g)$ *denotes the constant*

$$(1.12)\qquad \frac{(-1)^{(g-1)/2}\{\text{or }(-1)^{g/2}\}\cdot\Gamma((1/2)g)}{\sqrt{p_1p_2\cdots p_g}\,\pi^{(g-1)/2}\Gamma(g/2)}\varepsilon_1\cdots\varepsilon_g$$

according as g is odd or even, and $t'_\pm(\gamma,\gamma')$ *are also constants.*

2. It is well-known that the Green function $G(\gamma,\gamma'|z)=(z-A)^{-1}_{\gamma,\gamma'}$ for $\gamma,\gamma'\in\Gamma-\Omega$ can be described as follows:

$$(2.1)\qquad G(\gamma,\gamma'|z)=G_0(\gamma,\gamma'|z)-\sum_{\omega,\omega'\in\Omega}G_0(\gamma,\omega|z)H(\omega,\omega'|z)G_0(\omega',\gamma'|z)$$

where $(H(\omega,\omega'|z))_{\omega,\omega'\in\Omega}$ denotes the inverse of the Toeplitz matrix $T_\Omega=(G_0(\omega,\omega'|z))_{\omega,\omega'\in\Omega}$ of order $|\Omega|$, the number of elements of Ω. For $z\in \boldsymbol{C}-[-1,1]$, T_Ω is invertible. In fact, the symmetric bilinear form

$$(2.2)\qquad \Phi(u,v)=\sum_{\omega,\omega'\in\Omega}G_0(\omega,\omega'|z)u(\omega)v(\omega')$$

on $l^2(\Omega)$ has the definite real part for $z>1$ or $z<-1$ and the definite imaginary part for $\mathrm{I}_m z\neq 0$. For $\xi\in S^{g-1}$ such that $\xi_1\cdots\xi_g\neq 0$, we have the formula for the transmission coefficient $\alpha(\xi|z)$:

$$(2.3)\qquad \frac{1}{\alpha(\xi|z)}=\lim_{\gamma'\to\xi}\frac{G(e,\gamma'|z)}{G_0(e,\gamma'|z)}=1-\sum_{\omega,\omega'\in\Omega}G_0(e,\omega|z)H(\omega,\omega'|z)K_0(\omega',\xi|z)$$

and the basic eigenfunction

(2.4) $K(\gamma, \xi' | z) = \alpha(\xi' | z)\{K_0(\gamma, \xi' | z) - \sum_{\omega,\omega'} G_0(\gamma, \omega | z) H(\omega, \omega' | z) K_0(\omega', \xi' | z)\}.$

The asymptotic behaviour of $K(\gamma, \xi' | z)$ is as follows. For $\gamma \to \xi$,

(2.5) $$K(\gamma, \xi' | z) \sim \alpha(\xi | z)[K_0(\gamma, \xi' | z) + \beta(\xi, \xi' | z) G_0(\gamma, e | z)]$$

where $\beta(\xi, \xi' | z)$ denotes the scattering operator on S^{g-1}:

(2.6) $$\beta(\xi, \xi' | z) = -\sum_{\omega,\omega'} K_0(\omega, \xi | z) H(\omega, \omega' | z) K_0(\omega', \xi' | z).$$

Hence the determinant $S(z)$ of the matrix T_Ω plays the crucial role in the behaviour of $G(\gamma, \gamma' | z)$ and $\beta(\xi, \xi' | z)$ ([3]). We denote by T'_Ω the matrix of order $|\Omega|$ with entries $t'_\pm(\gamma, \gamma')$ for $\gamma, \gamma' \in \Omega$. Then

Lemma 2. (i) *$T_\Omega(\lambda \pm i0)$ is invertible for all $\lambda \in [-1, 1]$ if $g \geq 3$.*

(ii) *Assume $g=2$ and $\lambda = 2p_1\varepsilon_1 + 2p_2\varepsilon_2$, with $\varepsilon_1, \varepsilon_2 = \pm 1$. We denote by $T_\Omega^{(0)}$ the matrix of order $|\Omega|$ with entries $(-1)^{(m'_1 - m_1)\varepsilon_1 + (m'_2 - m_2)\varepsilon_2}$ for (m_1, m_2), $(m'_1, m'_2) \in \Omega$. Then the polynomial of h*

(2.7) $$\det(T_\Omega^{(0)} + hT'_\Omega) = h^{|\Omega|} \det T'_\Omega + h^{|\Omega|-1} \cdot \Delta_1(T'_\Omega, T_\Omega^{(0)})$$

does not vanish identically.

As an immediate consequence of it, we have

Proposition 2. *We fix $\lambda \in [-1, 1]$.*

i) *If $g \geq 3$, then $S(\lambda \pm i0)$ exists and is different from zero.*

ii) *If $g=2$, then $S(\lambda \pm i0)$ exists and is different from zero for $\lambda \neq 2p_1 \pm 2p_2$. Near $\lambda = 2p_1\varepsilon_1 + 2p_2\varepsilon_2$, we have*

(2.8) $$S(z) \sim C_0 \log(z - 2p_1\varepsilon_1 - 2p_2\varepsilon_2) + C_1$$

such that C_0 or C_1 is different from zero.

This gives us the following conclusion:

Theorem 1. *$G(\gamma, \gamma' | z)$ is holomorphic outside* $[-1, 1]$ *and has no poles along* $[-1, 1]$ *in* $I_m z \geq 0$ *or* $I_m z \leq 0$. *The operator A has no point spectrum.*

This is a difference analogue of the classical Rellich Theorem ([6]).

3. Let $\bar{\Gamma}$ be the compactification of Γ with the boundary S^{g-1}. Let $\mathfrak{k}$ be a cone in Γ with summit e and $\bar{\mathfrak{k}}$ be its closure in $\bar{\Gamma}$. The density matrix $\mu(d\xi | \lambda)$ is a Radon measure on S^{g-1} such that

(3.1) $$\lim_{\delta \downarrow 0} \frac{\delta}{\pi} \sum_{\gamma' \in \mathfrak{k}} |G(0, \gamma' | \lambda + i\delta)|^2 = \int_{\bar{\mathfrak{k}} \cap S^{g-1}} \mu(d\xi | \lambda).$$

We compute the left hand side for a special infinitesimal cone.

Because of symmetry property of $G(\gamma, \gamma' | z)$ we have only to compute $\mu(d\xi | \lambda)$ in the direction ξ such that $\xi_1 > 0, \cdots, \xi_g > 0$. We choose positive numbers $a_j, b_j, 2 \leq j \leq g$ such that $b_j - a_j$ are very small. We denote by $[a_2, \cdots, a_g; b_2, \cdots, b_g]$ a small cone $\mathfrak{k}$ in Γ consisting of elements $\gamma' = \sigma_1^{m'_1} \cdots \sigma_g^{m'_g}$ such that $a_j \leq m'_j / m'_1 \leq b_j$, $2 \leq j \leq g$. Since $G(e, \gamma' | z)$ has no poles along $[-1, 1]$, we have

(3.2) $$\lim_{\delta \downarrow 0} \frac{\delta}{\pi} |G(e, \gamma' | \lambda + i\delta)|^2 = 0.$$

Hence in view of (1.8) and (2.3)

(3.3) $$\lim_{\delta \downarrow 0} \frac{\delta}{\pi} \sum_{\gamma' \in [a_2, \cdots, a_g; b_2, \cdots, b_g]} |G(e, \gamma' | \lambda + i\delta)|^2$$

$$=\frac{1}{|\alpha(\xi\,|\,\lambda+i0)|^2}\lim_{\delta\downarrow 0}\frac{\delta}{\pi}\sum_{\gamma'\in[a_2,\cdots,a_g;b_2,\cdots,b_g]}|G_0(e,\gamma'\,|\,\lambda+i\delta)|^2$$

for arbitrary $\xi\in S^{g-1}$ such that $a_j\le\xi_j/\xi_1\le b_j$.

(1.8), (3.3) and an elementary computation imply

$$\mu(d\xi\,|\,\lambda)=\frac{1}{(2\pi)^g}\left|\frac{d\zeta_2\wedge\cdots\wedge d\zeta_g}{\zeta_1\prod_{j=2}^g\sqrt{\zeta_j^2+4p_j^2}}\right|\Big/|\alpha(\xi\,|\,\lambda+i0)|^2 \tag{3.4}$$

because

$$|\hat{\omega}_j|^2\sim 1-\frac{2\xi_j\delta}{\sqrt{4p_j^2+\zeta_j^2}}\Big/\left(\sum_{j=1}^g\frac{|\rho|\,\xi_j^2}{\sqrt{4p_j^2+\zeta_j^2}}\right)$$

through the substitution $\zeta_j=\rho\xi_j$. This enables us to give

Definition. The Radon measure $\mu(d\xi\,|\,\lambda)$ on S^{g-1} for $\lambda\in[-1,1]$ is defined by (3.5) on the image of κ from the real Fermi hypersurface $V_\lambda\cap\boldsymbol{R}^g$ and vanishes elsewhere. This is identified with the canonical form on $V_\lambda\cap\boldsymbol{R}^g$ by κ:

$$\kappa^*\mu(d\xi\,|\,\lambda)=\frac{1}{(2\pi)^g}\left[\frac{d\theta_1\wedge\cdots\wedge d\theta_g}{dF}\right]_{V_\lambda}\Big/|\alpha(\xi\,|\,\lambda+i0)|^2. \tag{3.5}$$

The formula of eigenfunction expansion can be stated as follows ([3]):

Theorem 2. *The spectral kernel $d\theta(\gamma,\gamma'\,|\,\lambda)$ is absolutely continuous for $\lambda\in[-1,1]$ and has the expression*

$$d\theta(\gamma,\gamma'\,|\,\lambda)=K(\gamma,\xi\,|\,\lambda+i0)K(\gamma',\xi\,|\,\lambda-i0)\cdot\mu(d\xi\,|\,\lambda)d\lambda. \tag{3.6}$$

The support of $\mu(d\xi\,|\,\lambda)$ coincides with the image of the Gauss map κ from $V_\lambda\cap\boldsymbol{R}^g$. Morse Theory shows that κ is not necessarily bijective unless $\max_j(1-4p_j)<|\lambda|<1$.

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