89. A Note on the Mean Value of the Zeta and L-functions. II

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1. In the present note we consider the mean square of *individual* Dirichlet L-functions.

Let χ be a *primitive* character (mod q), and put

$$E(T,\chi)=\int_0^T\left|L\left(\frac{1}{2}+it,\ \chi\right)\right|^2dt-\frac{\varphi(q)}{q}T\left\{\log(qT/2\pi)+2\gamma+2\sum_{p|q}(\log p)/(p-1)\right\},$$

where φ is the Euler function, γ the Euler constant, and p is a prime divisor of q. Then our problem is to find an estimate of $E(T,\chi)$ as uniform as possible for both parameters q and T. Our argument is based on the following χ-analogue of the important formula (3.4) of Atkinson [1].

Lemma 1. *If* $0<Re(u)<1$ *then*

$$(1)\qquad L(u,\chi)L(1-u,\bar{\chi})=\frac{\varphi(q)}{q}\left\{\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(u)+\frac{\Gamma'}{\Gamma}(1-u)\right)+2\gamma+\log\frac{q}{2\pi}+2\sum_{p|q}\frac{\log p}{p-1}\right\}+g(u,\chi)+g(1-u,\bar{\chi}),$$

where $g(u,\chi)$ *is the analytic continuation of*

$$(2)\qquad \sum_{n=1}^{\infty}a(n,\chi)\int_0^\infty\exp(-2\pi iny/q)y^{-u}(1+y)^{u-1}dy+\sum_{n=1}^{\infty}\overline{a(n,\bar{\chi})}\int_0^\infty\exp(2\pi iny/q)y^{-u}(1+y)^{u-1}dy,$$

which is convergent when $Re(u)<0$. *Here*

$$a(n,\chi)=q^{-1}\sum_{a|n}\sum_{m=1}^{q}\chi(m)\bar{\chi}(m+a)\exp(2\pi imn/aq).$$

This can be proved by a simple modification of our argument used in [6]. We denote by $g_1(u,\chi)$ the first sum of (2). To get an explicit representation of $g_1(u,\chi)$ which holds at least for $Re(u)<3/4$, we need some information on

$$A(x)=\sum_{n\le x}a(n,\chi).$$

To this end we put

$$F(s,\chi)=\sum_{n=1}^{\infty}a(n,\chi)n^{-s},$$

which is obviously convergent for $Re(s)>1$. Expressing $F(s,\chi)$ by a combination of Hurwitz zeta-functions, we get

Lemma 2. $F(s,\chi)$ *is entire, and when* $Re(s)<0$

$$F(s,\chi)=2(q\tau(\chi))^{-1}(2\pi/q)^{2(s-1)}\Gamma^2(1-s)\times\sum_{n=1}^{\infty}\chi(n)d(n)n^{s-1}(\chi(-1)\exp(-2\pi in/q)-\cos(\pi s)\exp(2\pi in/q)),$$

where τ is the Gauss sum, and d is the divisor function.

Then we may show, by a routine argument, a truncated form of the Voronoi type expansion of $A(x)$, which gives rise to

Lemma 3. *For any $X\geqq 1$*

$$\int_X^{2X}|A(x)|^2\,dx\ll X^{3/2}+(qX)^{1+\varepsilon}.$$

Now by the partial summation we have, for any half an odd integer N,

$$g_1(u,\chi)=\sum_{n\leqq N}a(n,\chi)h(n,u)-A(N)h(N,u)-\int_N^\infty A(x)\frac{\partial}{\partial x}h(x,u)dx,\tag{3}$$

where

$$h(x,u)=\int_0^\infty \exp(-2\pi ixy/q)y^{-u}(y+1)^{u-1}dy.\tag{4}$$

And Lemma 3 implies the convergence of the integral of (3) for $Re(u)<3/4$ (cf. [1, p. 359]), whence the required analytic continuation.

2. Now integrating the expression (1) on the segment $u=1/2+it$, $-T\leqq t\leqq T$, we get

$$E(T,\chi)=\mathrm{Im}\,\{E_1(T,\chi)+E_1(T,\bar{\chi})\}+O(1),$$

where

$$E_1(T,\chi)=\int_{1/2-iT}^{1/2+iT}g_1(u,\chi)du.$$

Then by an idea of Jutila [4] (see also Ivić [3, p. 476]) we have

Lemma 4.

$$E(T,\chi)\ll\operatorname*{Max}_V\operatorname*{Min}_G\left\{GL+G^{-1}\left|\int_{-\infty}^\infty E_1(V+u,\chi)\exp(-(u/G)^2)du\right|\right\},$$

where $L=\log qT$ and $L\leqq G\leqq VL^{-1}$, $T/2\leqq V\leqq 2T$.

To estimate this integral we use (3) with an M such that

$$qV/2\leqq M\leqq qV,\qquad A(M)\ll(qV)^{1/4}+q^{1/2}(qV)^{\varepsilon}.$$

Lemma 3 implies obviously the existence of such an M. Next in (4) we take the new path of integration: $y=r\exp(-i\alpha)$ $(0\leqq r<\infty)$ with a small $\alpha>0$. Then it is not difficult to see the absolute convergence of all relevant multiple integrals; we may perform the integration with respect to u inside the x- and y- integrals, and then restore the line of y- integral to the original one. In this way we get

$$\begin{aligned}(5)\quad &(2i\sqrt{\pi}G)^{-1}\int_{-\infty}^\infty E_1(V+u,\chi)\exp(-(u/G)^2)du\\ &=\sum_{n\leqq N}a(n,\chi)\int_0^\infty f_1(n,y)g(y)dy-A(N)\int_0^\infty f_1(N,y)g(y)dy\\ &\quad -V\int_N^\infty A(x)x^{-1}\int_0^\infty(1+y)^{-1}f_0(x,y)g(y)dydx+\int_N^\infty A(x)x^{-1}\int_0^\infty(1+y)^{-1}\\ &\quad\times\left(\frac{1}{2}+\frac{1}{2}G^2\log(1+1/y)+(\log(1+1/y))^{-1}\right)f_1(x,y)g(y)dydx\\ &=P_1-P_2-P_3+P_4,\end{aligned}$$

say, where

$$f_\nu(x,y)=\exp(-2\pi ixy/q)\cos(V\log(1+1/y)-\nu\pi/2),$$
$$g(y)=(y(y+1))^{-(1/2)}(\log(1+1/y))^{-1}\exp(-(1/4)(G\log(1+1/y))^2).$$

To estimate P_3 we divide it into two parts P_{31} and P_{32} according to $y \leqq qV/x$ and $y > qV/x$. Note that we have $qV \leqq x$. It is easy to see that P_{31} is negligible. As for P_{32} we have

$$P_{32} = -V \int_N^\infty A(x) x^{-1} \int_{qV/x}^\infty k(y)((1+y)^{-1} g(y))' dy dx,$$

where

$$k(y) = \int_{qV/x}^y f_0(x, \xi) d\xi.$$

The second mean value theorem gives $k(y) \ll q/x$; also we have $((1+y)^{-1}g(y))' \ll \exp(-G^2/20)$ if $y \leqq 1$ and $\ll (G^2 y^{-4} + y^{-2}) \exp(-(G/2y)^2)$ if $y \geqq 1$. These and Lemma 3 yield

(6) $$P_3 \ll ((qV)^{1/4} + q^{1/2}(qV)^{\varepsilon}) G^{-1}.$$

In just the same way we may show that

(7) $$P_4 \ll ((qV)^{1/4} + q^{1/2}(qV)^{\varepsilon}) V^{-1}$$

and

(8) $$P_2 \ll ((qV)^{1/4} + q^{1/2}(qV)^{\varepsilon}) V^{-1}.$$

3. P_1 is more difficult to estimate than other P's; the difficulty is caused by the fact that we now need a sharp estimate of individual $a(n, \chi)$. For this sake we appeal to

Lemma 5. *If q is a prime, then*

$$|a(n, \chi)| \leqq 2d(n)(q, n)^{1/2} q^{-(1/2)}.$$

This is a simple consequence of a result of Weil [7].
Thus we assume, hereafter, that our modulus q is a prime.
Now we put

$$l(x, y) = \int_{GL^{-1}}^y f_1(x, \xi) d\xi.$$

Then we have

$$P_1 = -\sum_{n \leqq N} a(n, \chi) \int_{GL^{-1}}^\infty l(n, y) g'(y) dy + O(e^{-L^2}).$$

We note that $g'(y) \ll G^2 y^{-3}$; also $l(n, y) \ll q/n$ if $n > qVL^2G^{-2}$ and $\ll y^{3/2} V^{-1/2}$ if $n \leqq qVL^2G^{-2}$. These and Lemma 5 yield

(9) $$P_1 \ll ((qV)^{1/2} G^{-(1/2)} + q^{1/2}) L^4.$$

Therefore from Lemma 4 and (5)–(9) we obtain

Theorem. *Let χ be a non-principal character* mod q, *a prime. Then we have, for $T \geqq 1$,*

$$E(T, \chi) \ll ((qT)^{1/3} + q^{1/2}) (\log qT)^4.$$

Remark 1. Our result should be compared with Theorem 2 of Heath-Brown [2].

Remark 2. In our later notes the χ-analogue of Atkinson's formula [1, p. 354] and the twelfth power moment of individual L-functions (cf. Meurman [5]) will be investigated by elaborating the above argument, both for composite moduli.

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