81. *Parametrices and Propagation of Singularities near Gliding Points for Mixed Problems for Symmetric Hyperbolic Systems. I*

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1. Introduction. Let $P(x, D)$ be a symmetric hyperbolic system defined on R^{n+1} in the form:

$$P(x, D)=\sum_{k=0}^{n} A_k(x)D_k+C(x), \qquad D_k=-i\partial/\partial x_k,$$

where $x=(x_0, x_1, \cdots, x_n)$, $A_k(x)$ are hermitian $m\times m$ matrices and $A_0(x)$ is positive definite. Consider the following mixed problem in the closed half space $X=\{x=(x', x_n)\,;\, x_n\geqq 0,\ x'=(x_0, x_1, \cdots, x_{n-1})\in X'=R^n\}$ with boundary ∂X:

(1) $P(x, D)u=0$ in X, $B(x)u=f$ on ∂X, $u(x)=0$ in $X\cap\{x_0\ll 0\}$.

In a previous paper [5] we proved, under certain restrictions on P and B, that there exists a parametrix near the diffractive point. The purpose of this note is to show the existence of a parametrix near the gliding point and study the propagation of singularities. Such results have been obtained by Eskin [2] and Petkov [8] in the case where the uniform Lopatinski condition (see (3) below) is satisfied, and by Taylor and Melrose [10] in a case, analogous to the Neumann problem for d'Alembertian where this condition is violated, which is a special case of ours. When $f=0$ but the initial data do not always vanish, the propagation of singularities has been investigated by several authors ([1], [3], [4] and [7]).

It seems that, to show the existence of a parametrix near a gliding point, one needs to make use of an Airy function $A_0(z)=Ai(-z)$ having zeros on the positive real axis. This causes difficulties which do not appear in the diffractive case. In fact, the boundary condition leads to an equation on the boundary which involves a Fourier integral operator with singular phase function. Moreover, when one solves the equation in a (not conic) region near the glancing surface, a pseudodifferential operator belonging to a bad class $OPS_{0,0}$ appears. Furthermore, if P is not strictly hyperbolic, in addition, if there exist two or more waves associated with a gliding ray, one can not reduce the equation to another which involves only one unknown.

As is seen in [2] or [9], the uniform Lopatinski condition guarantees that one can overcome these difficulties. Now suppose this is violated. In order to derive a basic a priori estimate for the equation on the boundary we then make an assumption, on the zeros of the Lopatinski determinant $R_0(x', \xi')$, which is more restrictive than condition (iv) in [5]. Besides, to assure the outgoing property we assume $R_0(x', \xi')$ does not vanish for $x_0 \ll 0$. In some cases, one can modify $B(x)$ so that this hypothesis is fulfilled. Suppose in addition there are such two or more waves as described above. We then assume also that $R_0(x', \xi')$ vanishes on the glancing surface for x_0 near 0. The main results in this note have been announced in [6], together with some applications.

2. Notations and assumptions. By $\xi = (\xi', \xi_n)$ we denote the covariables of $x = (x', x_n)$. Let $H^{\mathrm{loc}}_{(k,s)}(V)$ be the same Sobolev space as in [1], where k, s and V are a nonnegative integer, a real number and a relative open set in X, respectively. We then denote by $H^{\mathrm{loc}}_{\infty,-\infty}(V)$ the union of $\bigcap_{k=0}^{\infty} H^{\mathrm{loc}}_{(k,s_k)}(V)$ for all decreasing sequences $\{s_k\}_{k=0}^{\infty}$.

We assume P is of constant multiplicity. Then, denoting by $P_1(x, \xi)$ the principal symbol of P, one can write

$$\det P_1(x, \xi) = Q_1(x, \xi)^{m_1} \cdots Q_r(x, \xi)^{m_r} \tilde{Q}(x, \xi'),$$

where $Q_1, \cdots, Q_r$ and $\tilde{Q}$ are homogeneous polynomials in ξ which have no common zero in ξ_0; $Q_1, \cdots, Q_r$ are strictly hyperbolic with respect to ξ_0; and $\tilde{Q}$ is independent of ξ_n. We also suppose that ∂X is noncharacteristic for $Q_1, \cdots, Q_r$ and that, for each $(x', \xi') \in T^*X' \backslash 0$, the multiplicity of the real roots ξ_n of $Q(x', 0, \xi', \xi_n) = 0$ is at most double and there is at most one double real root, where Q is the product of Q_1 through Q_r. Let d^+ be the number of the positive eigenvalues of A_n. We then suppose B is a $d^+ \times m$ matrix of maximal rank. Besides, $A_k(x)$, $C(x)$ and $B(x)$ are assumed to be smooth (i.e., C^∞) and constant for $|x|$ large enough. Moreover we assume the boundary condition $Bu = 0$ is maximally dissipative.

Now, let $(\bar{x}', \bar{\xi}') \in T^*X' \backslash 0$ be a (fixed) gliding point (see [1]), by definition, a point such that for some j, say, $j = 1$, $Q_1(\bar{x}, \bar{\xi}', \xi_n) = 0$ has a double real root $\bar{\xi}_n$ and $\{Q_1, \partial Q_1/\partial \xi_n\}(\bar{x}, \bar{\xi})$ is negative, where $\bar{x} = (\bar{x}', 0) \in \partial X$, $\bar{\xi} = (\bar{\xi}', \bar{\xi}_n)$ and $\{\ ,\ \}$ denotes the Poisson bracket. In what follows we restrict ourselves to a conic neighborhood of $\iota^{*-1}(\bar{x}', \bar{\xi}')$, where ι^* is the pullback of $T^*X|_{\partial X}$ into T^*X' induced by the natural projection ι of X' into X such that $\iota(X') = \partial X$. Since $Q_1 = \partial Q_1/\partial \xi_n = 0$ and $\partial^2 Q_1/\partial \xi_n^2 \neq 0$ at $(\bar{x}, \bar{\xi})$, one can then write $Q_1 = Q_0 Q_1'$, where $Q_1'(\bar{x}, \bar{\xi}) \neq 0$ and

$$Q_0(x, \xi) = (\xi_n - \lambda(x, \xi'))^2 - \mu(x, \xi')$$

in a conic neighborhood of $(\bar{x}, \bar{\xi})$. Here $\lambda(x, \xi')$, $\mu(x, \xi')$ are real valued smooth functions, analytic and homogeneous in ξ' of degree 1, 2, respectively, such that $\mu(\bar{x}, \bar{\xi}') = 0$, $\lambda(\bar{x}, \bar{\xi}') = \bar{\xi}_n$ and

$$\{\xi_n - \lambda, \mu\}(x, \xi) < 0 \text{ when } \mu(x, \xi') = 0 \text{ and } x_n = 0. \tag{2}$$

A null bicharacteristic of $\mu(x', 0, \xi')$ is also called a gliding ray (or limiting bicharacteristic), which can be parametrized by x_0, because Q_1 is strictly hyperbolic and hence $\partial\mu/\partial\xi_0 \neq 0$. (See [1], [2].)

In order to define a Lopatinski determinant, let $W_0(x', \xi', \xi_n)$ be a smooth $m \times m_1$ matrix of maximal rank, homogeneous of degree 0 in ξ and analytic in ξ_n, which is a basis of $\ker P_1(x', 0, \xi', \xi_n)$ when $Q_0(x', 0, \xi', \xi_n)=0$. Besides, let $W_h^+(x', \xi')$ or $W_e^+(x', \xi')$ be, respectively, a smooth basis of the root subspace of $P_1(x', 0, \xi', \xi_n)$ corresponding to the outgoing simple real roots ξ_n of $(Q/Q_0)(x', 0, \xi', \xi_n)=0$ or to the outgoing non-real roots. Set

$$R(x', \xi', \xi_n)=\det B(x', 0)(W_0(x', \xi', \xi_n),\ W_h^+(x', \xi'),\ W_e^+(x', \xi')).$$

Moreover let $\xi_n^+(x', \xi')$ be the outgoing root of $Q_0(x', 0, \xi', \xi_n)=0$ and set $R_0(x', \xi')=R(x', \xi', \xi_n^+(x', \xi'))$, which is called a Lopatinski determinant. Then we say the uniform Lopatinski condition is satisfied at $(\bar{x}', \bar{\xi}')$ if

$$R_0(\bar{x}', \bar{\xi}') \neq 0. \tag{3}$$

When this is violated, we suppose

$$R_{\xi_n}(\bar{x}', \bar{\xi}', \bar{\xi}_n) \neq 0, \qquad \text{where } R_{\xi_n}=\partial R/\partial\xi_n, \tag{4}$$

and set $R_1(x', \xi')=(R/R_{\xi_n})(x', \xi', \lambda(x', 0, \xi'))$. We then assume the following three conditions are satisfied on the glancing surface

$$N_0=\{(x', \xi') \in T^*X' \backslash 0\,;\ \mu(x', 0, \xi')=0\}:$$

(H_1) There is a positive number $\delta_1<\pi/2$ such that $\arg R_1(x', \xi')$ is contained in the closed interval $[(\pi/2)+\delta_1, (3\pi/2)-\delta_1]$ or $[(-\pi/2)+\delta_1, (\pi/2)-\delta_1]$, according as $\partial\mu/\partial\xi_0$ is positive or negative.

(H_2) There is a positive number δ_2 such that $R_0(x', \xi')\neq 0$ for $x_0<\bar{x}_0-\delta_2$, where we have set $\bar{x}'=(\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_{n-1})$.

(H_3) When $m_1 \geqq 2$, there is a positive number $\delta_3<\delta_2$ such that $R_0(x', \xi')=0$ for $x_0>\bar{x}_0-\delta_3$.

3. Main results. For simplicity of description suppose $\bar{x}_0=0$. We then obtain the following.

Theorem[1]**.** *Assume* (H_1) *through* (H_3) *are satisfied on* N_0 *if* (3) *is violated. Let* f *be a distribution in* X' *with compact support such that* WF (f) *is contained in a small conic neighborhood of* $(\bar{x}', \bar{\xi}')$. *Then there exists a parametrix* $E(f) \in H^{\mathrm{loc}}_{\infty,-\infty}(X_T)$ *for* (1) *such that* $PE(f) \in C^\infty(X_T)$, $BE(f)|_{x_n=0}-f \in C^\infty(X'_T)$ *and* $E(f) \in C^\infty(X \cap \{x_0 \ll 0\})$, *where* T *is a positive number*, $X_T=X \cap \{x_0<T\}$ *and* $X'_T=X' \cap \{x_0<T\}$. *Moreover* $E(f)|_{X_T}$ *is smooth up to the boundary at each point* $(x', \xi') \in T^*X' \backslash 0$ *in the complement of*

$$\mathrm{WF}\,(f) \cup M_0^+(f) \cup (\textstyle\bigcup_{k=0}^{\infty} \phi_+^k(\mathrm{WF}(f) \cap N_+)). \tag{5}$$

Here $M_0^+(f)$ *is the union of all gliding rays which start from* WF (f) $\cap N_0$ *and go into the positive* x_0 *direction*, $N_+=\{(x', \xi') \in T^*X' \backslash 0\,;\ \mu(x', 0, \xi')>0\}$, *and* ϕ_+ *is the canonical transformation on* N_+ *such that the outgoing bicharacteristic of* Q_0 *starting from* $\iota^{*-1}(x', \xi') \cap Q_0^{-1}(0)$

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[1] The proof of this theorem will be published in the next Proceedings.

intersects $T^*X|_{\partial X}$ *at* $\iota^{*-1}(\phi_+(x',\xi'))\cap Q_0^{-1}(0)$ *once more. Besides,* ϕ_+^k *denotes the k-th power of* ϕ_+.

For the definition of "smooth up to the boundary" see [7], p. 595. On the propagation of singularities we have the followings, the latter of which can be reduced to the former.

Corollary 1. *Let the hypotheses in Theorem be fulfilled and let* T *be such as above. Let* $u\in H^{\mathrm{loc}}_{(0,s)}(X_T)$ *for some* $s\in R^1$. *Suppose* $Pu\in C^\infty(X_T)$, $u\in C^\infty(X\cap\{x_0\ll 0\})$ *and* $Bu|_{x_n=0}-f\in C^\infty(X'_T)$. *Then* $u|_{X_T}$ *is smooth up to the boundary at each* $(x',\xi')\in T^*X'\setminus 0$ *outside* (5).

Corollary 2. *Assume* (H_1) *through* (H_3) *are satisfied on* N_0 *if* (3) *is violated. Let* $u\in H^{\mathrm{loc}}_{\infty,-\infty}(V)$ *for a neighborhood* V *of* $\bar{x}$. *Assume* $Pu\in C^\infty(V)$, $(\bar{x}',\bar{\xi}')\notin \mathrm{WF}(Bu|_{x_n=0})$ *and* $\mathrm{WF}(u|_{x_n=0})\cap\Gamma(\bar{x}',\bar{\xi}')\cap\{-\delta<x_0<0\}=\phi$ *for some* $\delta>0$, *where* $\Gamma(\bar{x}',\bar{\xi}')$ *is the gliding ray through* $(\bar{x}',\bar{\xi}')$. *Besides, suppose* $\mathrm{WF}(u|_{V\setminus\partial X})$ *intersects no incoming null bicharacteristic of* Q/Q_0 *which arrives at* $\iota^{*-1}(\bar{x}',\bar{\xi}')$. *Then* u *is smooth up to the boundary at* $(\bar{x}',\bar{\xi}')$.

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