50. *On the Euler-Poisson-Darboux Equation and the Toda Equation. II*

By Yoshinori KAMETAKA
Department of Applied Mathematics, Faculty of Engineering, Ehime University

(Communicated by Kôsaku YOSIDA, M. J. A., June 12, 1984)

§1. **Summary.** This note is a sequel to the preceding paper [1]. We can construct bases of some special class of solutions of the Toda equation with two time variables

(1.1) $XY\log t_n=t_{n+1}t_{n-1}/t_n^2 \qquad (X=\partial/\partial x,\ Y=\partial/\partial y,\ t_n=t_n(x,y))$

using Gauss hypergeometric functions. By eigenfunction expansion we can construct solutions which can be expressed by hypergeometric functions with two variables of order two. According to the Horn's list ([2]) F_1, F_2, F_3 (Appell hypergeometric functions) G_2, H_2, H_4, Φ_1, Ψ_1, Ξ_1, Ξ_2, Γ_1, $\boldsymbol{H}_2$, $\boldsymbol{H}_3$ and $\boldsymbol{H}_{11}$ appear.

By quadratic transformations for Gauss hypergeometric functions we also have rational solutions given by Gegenbauer, Legendre and Chebyshev polynomials.

§2. **Connection formulas for Gauss hypergeometric solutions.** In our previous work ([1]) we showed that

(2.1) $$\begin{aligned} &f_n(\alpha,\beta,\gamma;x,y)\\ &\quad=(\gamma+1-\alpha-\beta)_n(y-x)^{\beta-n}y^{\gamma-\beta}F(\beta-\gamma,\beta-n,\alpha+\beta-\gamma-n;x/y),\\ &g_n(\alpha,\beta,\gamma;x,y)\\ &\quad=((1-\alpha)_n(1-\beta)_n/(\gamma+2-\alpha-\beta)_n)(y-x)^{\beta-n}x^{n+1+\gamma-\alpha-\beta}y^{\alpha-1-n}\\ &\qquad\times F(n+1-\alpha,\gamma+1-\alpha,n+2+\gamma-\alpha-\beta;x/y)\end{aligned}$$

are bases of the 2-dimensional vector space $T\cap\{u_n\in\ker(C_n-\gamma)\}$. Another bases are possible.

Theorem 2.1. *Put*

(2.2) $$\begin{aligned} &f_n^{(1)}(\alpha,\beta,\gamma;x,y)\\ &\quad=(1-\beta)_n(y-x)^{\alpha-n}y^{\gamma-\alpha}F(\alpha-n,\alpha-\gamma,1+\alpha-\beta;1-x/y).\end{aligned}$$

$f_n^{(1)}(\alpha,\beta,\gamma;x,y)$ and $f_n^{(1)}(\beta,\alpha,\gamma;x,y)$ are bases of the vector space $T\cap\{u_n\in\ker(C_n-\gamma)\}$. We have the following relations.

(2.3) $$\begin{aligned} &(-A)^k f_n^{(1)}(\alpha,\beta,\gamma;x,y)=(\alpha-\gamma)_k f_n^{(1)}(\alpha,\beta,\gamma-k;x,y),\\ &B_n^k f_n^{(1)}(\alpha,\beta,\gamma;x,y)=(\gamma+1-\beta)_k f_n^{(1)}(\alpha,\beta,\gamma+k;x,y).\end{aligned}$$

Theorem 2.2. *When $|\arg(1-x/y)|<\pi$ we have the following connection formula*

(2.4) $$\begin{pmatrix} f_n(\alpha,\beta,\gamma;x,y)\\ g_n(\alpha,\beta,\gamma;x,y)\end{pmatrix}=\begin{pmatrix} K(\alpha,\beta,\gamma) & K(\beta,\alpha,\gamma)\\ L(\alpha,\beta,\gamma) & L(\beta,\alpha,\gamma)\end{pmatrix}\begin{pmatrix} f_n^{(1)}(\alpha,\beta,\gamma;x,y)\\ f_n^{(1)}(\beta,\alpha,\gamma;x,y)\end{pmatrix}$$

where

$$K(\alpha,\beta,\gamma)=\frac{\Gamma(\beta-\alpha)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\beta)\Gamma(\beta-\gamma)},\qquad L(\alpha,\beta,\gamma)=\frac{\Gamma(\gamma+2-\alpha-\beta)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\gamma+1-\alpha)}.$$

§3. Hypergeometric solutions. If

(3.1) $u_n=\sum_{k=0}^{\infty} a_k f_n(\alpha,\beta,\gamma-\varepsilon k\,;x,y)$ (ε is an integer)

converges then it belongs to T. If we choose ε and a_k suitably then we can express u_n by hypergeometric functions with two variables of order 2 which we can find in Horn's list ([2]).

Theorem 3.1 (Hypergeometric solutions). *α, β, γ, α', β' and ε' represent arbitrary constants. Put*

(3.2) $U_n(x,y)=(\gamma+1-\alpha-\beta)_n(y-x)^{\beta-n}y^{\gamma-\beta}$, $\quad V_n(y)=U_n(0,y)$.

(i) $\varepsilon=1$, $a_k=(\alpha')_k(\beta')_k/(\alpha+\beta-\gamma)_k k\,!$,

(3.3)
$$u_n=V_n(y)\sum_{j,k}\frac{(\alpha-n)_j(\beta-n)_j(\alpha')_k(\beta')_k}{(\alpha+\beta-\gamma-n)_{j+k}j\,!\,k\,!}(x/(x-y))^j(1/y)^k$$
$$=V_n(y)F_3(\alpha-n,\alpha',\beta-n,\beta',\alpha+\beta-\gamma-n\,;x/(x-y),1/y)$$
$$={}_2F_2(\alpha',\beta',\alpha-\gamma,\beta-\gamma\,;-A)f_n(\alpha,\beta,\gamma\,;x,y),$$

(3.4) $u_n(0,y)=V_n(y){}_2F_1(\alpha',\beta',\alpha+\beta-\gamma-n\,;1/y)$.

Especially if we take $\alpha'=\beta-\gamma$ then we have

(3.5)
$$u_n=U_n(x,y)\sum_{j,k}\frac{(\beta-\gamma)_{j+k}(\beta-n)_j(\beta')_k}{(\alpha+\beta-\gamma-n)_{j+k}j\,!\,k\,!}(x/y)^j(1/y)^k$$
$$=U_n(x,y)F_1(\beta-\gamma,\beta-n,\beta',\alpha+\beta-\gamma-n\,;x/y,1/y)$$
$$={}_1F_1(\beta',\alpha-\gamma\,;-A)f_n(\alpha,\beta,\gamma\,;x,y),$$

(3.6) $u_n(0,y)=V_n(y){}_2F_1(\beta-\gamma,\beta',\alpha+\beta-\gamma-n\,;1/y)$.

(ii) $\varepsilon=1$, $a_k=(\alpha')_k/(\alpha+\beta-\gamma)_k k\,!$,

(3.7)
$$u_n=V_n(y)\Xi_1(\alpha-n,\alpha',\beta-n,\alpha+\beta-\gamma-n\,;x/(x-y),1/y)$$
$$={}_1F_2(\alpha',\alpha-\gamma,\beta-\gamma\,;-A)f_n(\alpha,\beta,\gamma\,;x,y),$$

(3.8) $u_n(0,y)=V_n(y){}_1F_1(\alpha',\alpha+\beta-\gamma-n\,;1/y)$.

If $\alpha'=\beta-\gamma$ then we have

(3.9)
$$u_n=U_n(x,y)\Phi_1(\beta-\gamma,\beta-n,\alpha+\beta-\gamma-n\,;1/y,x/y)$$
$$={}_0F_1(\alpha-\gamma\,;-A)f_n(\alpha,\beta,\gamma\,;x,y),$$

(3.10) $u_n(0,y)=V_n(y){}_1F_1(\beta-\gamma,\alpha+\beta-\gamma-n\,;1/y)$.

(iii) $\varepsilon=1$, $a_k=1/(\alpha+\beta-\gamma)_k k\,!$,

(3.11)
$$u_n=V_n(y)\Xi_2(\alpha-n,\beta-n,\alpha+\beta-\gamma-n\,;x/(x-y),1/y)$$
$$={}_0F_2(\alpha-\gamma,\beta-\gamma\,;-A)f_n(\alpha,\beta,\gamma\,;x,y),$$

(3.12) $u_n(0,y)=V_n(y){}_0F_1(\alpha+\beta-\gamma-n\,;1/y)$.

(iv) $\varepsilon=-1$, $a_k=(\beta')_k(\gamma+1-\alpha-\beta)_k/(\varepsilon')_k k\,!$,

(3.13)
$$u_n=V_n(y)\sum_{j,k}\frac{(n+1+\gamma-\alpha-\beta)_{k-j}(\beta')_k(\alpha-n)_j(\beta-n)_j}{(\varepsilon')_k j\,!\,k\,!}$$
$$\times y^k(x/(y-x))^j$$
$$=V_n(y)H_2(n+1+\gamma-\alpha-\beta,\beta',\alpha-n,\beta-n,\varepsilon'\,;y,x/(y-x))$$
$$={}_1F_1(\beta',\varepsilon'\,;B_n)f_n(\alpha,\beta,\gamma\,;x,y),$$

(3.14) $u_n(0,y)=V_n(y){}_2F_1(n+1+\gamma-\alpha-\beta,\beta',\varepsilon'\,;y)$.

If $\varepsilon'=\gamma+1-\beta$ *then we have*

(3.15) $$u_n=U_n(x,y)\sum_{j,k}\frac{(\beta-n)_j(\beta')_k(n+1+\gamma-\alpha-\beta)_{k-j}(\beta-\gamma)_{j-k}}{j!k!}$$
$$\times(-x/y)^j(-y)^k$$
$$=U_n(x,y)G_2(\beta-n,\beta',n+1+\gamma-\alpha-\beta,\beta-\gamma;-x/y,-y)$$
$$={}_1F_1(\beta',\gamma+1-\beta;B_n)f_n(\alpha,\beta,\gamma;x,y),$$

(3.16) $$u_n(0,y)=V_n(y)_2F_1(\beta',n+1+\gamma-\alpha-\beta,\gamma+1-\beta;y).$$

(v) $\varepsilon=-1$, $a_k=(\gamma+1-\alpha-\beta)_k/(\varepsilon')_k k!$,

(3.17) $$u_n=V_n(y)H_{11}(n+1+\gamma-\alpha-\beta,\alpha-n,\beta-n,\varepsilon';y,x/(y-x))$$
$$={}_0F_1(\varepsilon';B_n)f_n(\alpha,\beta,\gamma;x,y),$$

(3.18) $$u_n(0,y)=V_n(y)_1F_1(n+1+\gamma-\alpha-\beta,\varepsilon';y).$$

If $\varepsilon'=\gamma+1-\beta$ *then we have*

(3.19) $$u_n=U_n(x,y)\Gamma_1(\beta-n,n+1+\gamma-\alpha-\beta,\beta-\gamma;-x/y,-y)$$
$$={}_0F_1(\gamma+1-\beta;B_n)f_n(\alpha,\beta,\gamma;x,y),$$

(3.20) $$u_n(0,y)=V_n(y)_1F_1(n+1+\gamma-\alpha-\beta,\gamma+1-\beta;y).$$

If

(3.21) $$u_n=\sum_{k=0}^{\infty}a_kf_n^{(1)}(\alpha,\beta,\gamma-\varepsilon k;x,y)\qquad(\varepsilon\text{ is an integer})$$

converges it belongs to T.

Theorem 3.2 (Hypergeometric solutions). α, β, γ, β', γ' *and* δ' *represent arbitrary constants. Put*

(3.22) $$\tilde{U}_n(x,y)=(1-\beta)_n(y-x)^{\alpha-n}y^{\gamma-\alpha},\qquad \tilde{V}_n(y)=(1-\beta)_ny^{\gamma-\alpha}.$$

(i) $\varepsilon=1$, $a_k=(\alpha-\gamma)_k(\beta')_k/(\gamma')_k k!$,

(3.23) $$u_n=\tilde{U}_n(x,y)\sum_{j,k}\frac{(\alpha-\gamma)_{j+k}(\alpha-n)_j(\beta')_k}{(1+\alpha-\beta)_j(\gamma')_kj!k!}(1-x/y)^j(1/y)^k$$
$$=\tilde{U}_n(x,y)F_2(\alpha-\gamma,\alpha-n,\beta',1+\alpha-\beta,\gamma';1-x/y,1/y)$$
$$={}_1F_1(\beta',\gamma';-A)f_n^{(1)}(\alpha,\beta,\gamma;x,y),$$

(3.24) $$(y-x)^{n-\alpha}u_n|_{x=y}=\tilde{V}_n(y)\,{}_2F_1(\alpha-\gamma,\beta',\gamma';1/y).$$

(ii) $\varepsilon=1$, $a_k=(\alpha-\gamma)_k/(\gamma')_k k!$,

(3.25) $$u_n=\tilde{U}_n(x,y)\Psi_1(\alpha-\gamma,\alpha-n,1+\alpha-\beta,\gamma';1-x/y,1/y)$$
$$={}_0F_1(\gamma';-A)f_n^{(1)}(\alpha,\beta,\gamma;x,y),$$

(3.26) $$(y-x)^{n-\alpha}u_n|_{x=y}=\tilde{V}_n(y)_1F_1(\alpha-\gamma,\gamma';1/y).$$

(iii) $\varepsilon=-1$, $a_k=(\gamma')_k(\delta')_k/(\gamma+1-\alpha)_k k!$,

(3.27) $$u_n=\tilde{U}_n(x,y)H_2(\alpha-\gamma,\alpha-n,\gamma',\delta',1+\alpha-\beta;1-x/y,-y)$$
$$={}_2F_2(\gamma',\delta',\gamma+1-\alpha,\gamma+1-\beta;B_n)f_n^{(1)}(\alpha,\beta,\gamma;x,y),$$

(3.28) $$(y-x)^{n-\alpha}u_n|_{x=y}=\tilde{V}_n(y)\,{}_2F_1(\gamma',\delta',\gamma+1-\alpha;y).$$

(iv) $\varepsilon=-1$, $a_k=(\gamma')_k/(\gamma+1-\alpha)_k k!$,

(3.29) $$u_n=\tilde{U}_n(x,y)H_2(\alpha-\gamma,\alpha-n,\gamma',1+\alpha-\beta;1-x/y,-y)$$
$$={}_1F_2(\gamma',\gamma+1-\alpha,\gamma+1-\beta;B_n)f_n^{(1)}(\alpha,\beta,\gamma;x,y),$$

(3.30) $$(y-x)^{n-\alpha}u_n|_{x=y}=\tilde{V}_n(y)_1F_1(\gamma',\gamma+1-\alpha;y).$$

(v) $\varepsilon=-1$, $a_k=1/(\gamma+1-\alpha)_k k!$,

(3.31) $$u_n=U_n(x,y)H_3(\alpha-\gamma,\alpha-n,1+\alpha-\beta;1-x/y,-y)$$
$$={}_0F_2(\gamma+1-\alpha,\gamma+1-\beta;B_n)f_n^{(1)}(\alpha,\beta,\gamma;x,y),$$

(3.32) $$(y-x)^{n-\alpha}u_n|_{x=y}=\tilde{V}_n(y)\,{}_0F_1(\gamma+1-\alpha\,;\,y).$$

(vi) $\varepsilon=2$, $a_k=(\alpha-\gamma)_{2k}/(\gamma')_k k!$,

(3.33) $$u_n=\tilde{U}_n(x,y)H_4(\alpha-\gamma,\alpha-n,\gamma',1+\alpha-\beta\,;\,1/y^2,1-x/y)$$
$$={}_0F_1(\gamma'\,;\,A^2)f_n^{(1)}(\alpha,\beta,\gamma\,;\,x,y),$$

(3.34) $$(y-x)^{n-\alpha}u_n|_{x=y}=\tilde{V}_n(y)\,{}_2F_1((\alpha-\gamma)/2,(\alpha-\gamma+1)/2,\gamma'\,;\,4/y^2).$$

§4. Quadratic transformations. By quadratic transformation for hypergeometric functions we have the following results.

Theorem 4.1 (Gegenbauer polynomial solutions).

(4.1) $$f_n^{(1)}(0,\beta,(\beta-1)/2\,;\,x,y)$$
$$=n!\,(y-x)^{-n}x^{n/2}y^{(\beta-1-n)/2}C_n^{((1-\beta)/2)}((\sqrt{x/y}+\sqrt{y/x})/2),$$

(4.2) $$f_n^{(1)}(0,0,-1/2\,;\,x,y)$$
$$=n!\,(y-x)^{-n}x^{n/2}y^{-(1+n)/2}P_n((\sqrt{x/y}+\sqrt{y/x})/2),$$

(4.3) $$\lim_{\beta\to1}\Gamma(1-\beta)f_n^{(1)}(0,\beta,(\beta-1)/2\,;\,x,y)$$
$$=(n-1)!\,(y-x)^{-n}x^{n/2}y^{-n/2}T_n((\sqrt{x/y}+\sqrt{y/x})/2).$$

$C_n^{(a)}(z)$, $P_n(z)$ and $T_n(z)$ are Gegenbauer, Legendre and Chebyshev polynomials respectively. These polynomials give rational solutions of the Toda equation in half space ($n\geqslant 0$ or $n\geqslant 1$).

References

[1] Y. Kametaka: On the Euler-Poisson-Darboux equation and the Toda equation. I. Proc. Japan Acad., **60**A, 145–148 (1984).

[2] A. Erdelyi *et al.*: Higher Transcendental Functions. vol. 1. McGraw-Hill, pp. 224–227 (1953).

[3] Y. Kametaka: On the telegraph equation and the Toda equation. Proc. Japan Acad., **60**A, 79–81 (1984).

[4] ——: On the confluent Euler-Poisson-Darboux equation and the Toda equation. Proc. Japan Acad., **60**A, 121–124 (1984).