46. A Note on M-Projective Modules. II

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In the previous note [7], we characterized "weakly" M-projective modules over a left PF ring. In the present note, we study another type of "weakly" M-projective modules with a finiteness condition.

1. Definition. Throughout, R denotes a ring with $1\neq 0$ and M a unital left R-module. Azumaya [2] introduced the notions of M-epimorphisms and M-projective modules. Now we add to these some finiteness restrictions.

Let A and B be left R-modules and g an epimorphism of A onto B. Then g is called a (*finite*) *M-epimorphism* iff there exists a homomorphism h of A into M such that $\mathrm{Ker}\, g\cap \mathrm{Ker}\, h=0$ (and the factor module $M/(\mathrm{Ker}\, g)h$ is finitely cogenerated). A left R-module P is called (*finitely*) *M-projective* iff for every homomorphism f of P into N and every epimorphism g of M onto N, with N an arbitrary (finitely cogenerated) left R-module, there exists a homomorphism h of P into M such that $hg=f$. Then we have the following immediately:

(1) Every finite M-epimorphism is an M-epimorphism and the converse holds if M is Artinian.

(2) Every M-projective module is finitely M-projective and the converse holds if M is Artinian.

(3) Every direct summand of a finitely M-projective module is finitely M-projective and every direct sum of finitely M-projective modules is finitely M-projective.

2. Characterization. The following is the dual of Theorem 4.1 in Ramamurthi and Rangaswamy [6] which can be proved analogously to the case of M-projective modules.

Proposition 1. *Let P be a left R-module. Then the following conditions are equivalent:*

(1) *P is finitely M-projective.*

(2) *For every homomorphism f of P into B and every finite M-epimorphism g of A onto B, with A and B left R-modules, there exists a homomorphism h of P into A such that $hg=f$.*

(3) *Every finite M-epimorphism onto P splits.*

3. The first result. A ring R is called *left co-Noetherian* iff each factor module of every finitely cogenerated left R-module is finitely cogenerated. For other characterizations, see [8], [3], [4] and

[1; p. 217].

Lemma 1 (See [7]). *Let R be a QF ring and P a finitely generated left R-module. If P is $\bar{M}$-projective for every finitely cogenerated factor module $\bar{M}$ of M, then P is M-projective.*

Proposition 2. *Let R be a left co-Noetherian QF ring and P a finitely generated left R-module. Then P is M-projective if P is finitely M-projective.*

Proof. Let $\bar{M}$ be a finitely cogenerated factor module of M. Then every homomorphic image of $\bar{M}$ is finitely cogenerated and therefore P is $\bar{M}$-projective since P is finitely M-projective. Thus, by Lemma 1, P is M-projective.

4. The second result. We need a lemma.

Lemma 2 (See [5; Lemma 1.1]). *Let x be a non-zero element in M. Then there exists a submodule B of M such that the factor module M/B is finitely cogenerated and B does not contain x.*

Theorem. *Let P be an (R, S)-bimodule: $P={}_RP_S$, where S is the endomorphism ring of ${}_RP$. Assume that the left S-module ${}_SP^*:=\mathrm{Hom}_R(P, M)$ is linearly compact. Then P is M-projective if P is finitely M-projective.*

Proof. Let A be a proper submodule of M and f a homomorphism of P into M/A, π the natural epimorphism of M onto M/A. For each x in M–A, there exists a submodule B_x of M such that M/B_x is finitely cogenerated and B_x includes A but does not contain x, by making use of Lemma 2. Let π_x be the natural epimorphism of M/A onto M/B_x. Then, since P is finitely M-projective, there exists a homomorphism g_x in P^* such that $f\pi_x=g_x\pi\pi_x$. Let $P^*(x)$ be an S-submodule of ${}_SP^*$ consisting of those homomorphisms h in P^* such that $Ph\subset B_x$. Now, for all x in M–A, the pairs of $P^*(x)$ and g_x make a finitely solvable system in P^*. It is shown as follows. For the intersection, say ${}_RB$, of B_{x_i} with $x_1, x_2, \cdots, x_n$ in M–A, M/B is finitely cogenerated since it is embedded in the direct sum of M/B_{x_i}. Then, since P is finitely M-projective, there exists a homomorphism g' in P^* such that $f\pi'=g'\pi\pi'$, where π' is the natural epimorphism of M/A onto M/B. Thus, both $P(f-g_{x_i}\pi)$ and $P(g'\pi-f)$ are included in B_{x_i}/A for each i. Accordingly $g'-g_{x_i}$ is in $P^*(x_i)$ for each i. Therefore, the system has a solution g in P^* since ${}_SP^*$ is linearly compact and hence $g-g_x$ is in $P^*(x)$ for each x in M–A. This deduces that $P(g\pi-f)$ is included in B_x/A for each x in M–A and so we have $g\pi=f$, as required.

Corollary. *Let P be an (R, S)-bimodule: $P={}_RP_S$, where S is the endomorphism ring of ${}_RP$. Assume that S is a left linearly compact ring. Then ${}_RP$ is P-projective* (i.e., *quasi-projective*) *if ${}_RP$ is finitely P-projective.*

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