131. *Analytic Singularities of Solutions of the Hyperbolic Cauchy Problem*

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1. Introduction. Kashiwara and Kawai [6] constructed fundamental solutions for (partially) micro-hyperbolic operators, micro-localizing the results of Bony and Schapira [3]. Miwa [8] applied their results and studied the propagation of analytic singularities (see, also, Bony-Schapira [4], Bony [2], Kashiwara-Schapira [7], Sjöstrand [10]). On the other hand, in [13], we micro-localized the results in Bronshtein [5] and studied singularities (in Gevrey classes) solutions of the Cauchy problem, using generalized Hamilton flows defined in [11]. So, applying the arguments in [13], we can easily obtain a result on analytic singularities of solutions of the hyperbolic Cauchy problem from the results of Kashiwara and Kawai [6].

2. Assumptions and results. Let $P(x,\partial/\partial x)=\sum_{|\alpha|\leqq m} a_\alpha(x)(\partial/\partial x)^\alpha$ be a partial differential operator, where $x=(x_1,\cdots,x_n)\in \boldsymbol{R}^n$ and $\partial/\partial x=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$. Assume that

(A-1) the $a_\alpha(x)$ are real analytic on $\boldsymbol{R}^n$,

(A-2) $P_m(x,\xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is hyperbolic with respect to $\vartheta=(1,0,\cdots,0)\in \boldsymbol{R}^n$, i.e.,

$$P_m(x,\sqrt{-1}\xi+\tau\vartheta)\neq 0 \qquad \text{for } x\in \boldsymbol{R}^n,\quad \xi\in \boldsymbol{R}^n \text{ and } \tau>0.$$

First let us define the localization $P_{mz}(\delta z)$ at $z=(x,\sqrt{-1}\xi)\in\sqrt{-1}T^*\boldsymbol{R}^n \cong R^n\times\sqrt{-1}\boldsymbol{R}^n$ by

$$P_m(x+s\delta x,\sqrt{-1}\xi+\sqrt{-1}s\delta\xi)=s^\mu(P_{mz}(\delta x,\delta\xi)+o(1)) \quad \text{as} \quad s\longrightarrow 0,$$

where $P_{mz}(\delta z)\not\equiv 0$ (in δz) is (homogeneous) polynomial of $\delta z=(\delta x,\delta\xi)\in T_z(\sqrt{-1}T^*\boldsymbol{R}^n)\cong \boldsymbol{R}^{2n}$. Then $P_{mz}(\delta z)$ is hyperbolic with respect to $(0,\vartheta)\in \boldsymbol{R}^{2n}$ (see [11]). Therefore, we can define $\Gamma(P_{mz},(0,\vartheta))$ as the connected component of the set $\{\delta z\in T_z(\sqrt{-1}T^*\boldsymbol{R}^n);\ P_{mz}(\delta z)\neq 0\}$ which contains $(0,\vartheta)$. Define

$$\Gamma_z=\Gamma(P_{mz},(0,\vartheta))\subset T_z(\sqrt{-1}T^*\boldsymbol{R}^n),$$

$$\Gamma_z^\sigma=\{(\delta x,\delta\xi)\in T_z(\sqrt{-1}T^*R^n);\ \delta x\cdot\delta\eta-\delta y\cdot\delta\xi\ (=\sigma((\delta x,\delta\xi),(\delta y,\delta\eta))) \geqq 0 \text{ for any } (\delta y,\delta\eta)\in\Gamma_z\},$$

$$K_z^\pm=\{z(t)\in\sqrt{-1}T^*\boldsymbol{R}^n;\ \{z(t)\} \text{ is a Lipschitz continuous curve satisfying } (d/dt)z(t)\in\Gamma^\sigma_{z(t)} \text{ (a.e. } t) \text{ and } z(0)=z, \text{ and } \pm t\geqq 0\}$$

(see [11]–[13]). It is easy to see that $K^\pm_{(x,0)}=K^\pm_x\times\{0\}$, where $K_x^\pm=\{x(t);\ \{x(t)\}$ is a Lipschitz continuous curve satisfying $(d/dt)x(t)\in\Gamma(P_m(x(t),\cdot),\vartheta)^*$ (a.e. t) and $x(0)=x$, and $\pm t\geqq 0\}$ and $\Gamma^*=\{\delta x;\ \delta x\cdot\delta\xi\geqq 0$ for any

$\delta\xi\in\Gamma\}$. We refer to Wakabayashi [13] for some properties of "flows" $K_z^\pm$. We denote by $\mathcal{B}(\boldsymbol{R}^n)$ the space of all hyperfunctions on $\boldsymbol{R}^n$. Define

$$\widehat{\mathrm{S.S.}}\,u=\{(x,\sqrt{-1}\xi)\in\sqrt{-1}T^*\boldsymbol{R}^n\,;\,x\in\operatorname{Supp}u\text{ and }\xi=0\}$$
$$\cup\{(x,\sqrt{-1}\xi)\in\sqrt{-1}T^*\boldsymbol{R}^n\,;\,\xi\neq 0\text{ and }(x,\sqrt{-1}\xi\infty)\in\mathrm{S.S.}\,u\}$$

for $u\in\mathcal{B}(\boldsymbol{R}^n)$, where S.S. u denotes the singular spectrum of u (see [9]). In order to state the result globally, we assume that

(A-3) $K_x^-\cap\{x_1\geqq 0\}$ is bounded for every $x\in\boldsymbol{R}^n$.

Theorem. *Assume that* (A-1)–(A-3) *are satisfied. If* $u\in\mathcal{B}(\boldsymbol{R}^n)$ *satisfies the Cauchy problem*

$$\text{(CP)}\qquad\begin{cases}P(x,\partial/\partial x)u=f,\\ \operatorname{Supp}u\subset\{x_1\geqq 0\},\end{cases}$$

where $f\in\mathcal{B}(\boldsymbol{R}^n)$ *and* $\operatorname{Supp}f\subset\{x_1\geqq 0\}$, *then*

$$\widehat{\mathrm{S.S.}}\,u\subset\{z\in\sqrt{-1}T^*\boldsymbol{R}^n\,;\,z\in K_w^+\text{ for some }w\in\widehat{\mathrm{S.S.}}\,f\}.$$

Remark. Bony and Schapira [3] proved well-posedness of the hyperbolic Cauchy problem in the framework of hyperfunctions.

3. Proof of theorem. We write

$$\hat{L}=\sqrt{-1}T^*\boldsymbol{R}^n\backslash\boldsymbol{R}^n\cong\boldsymbol{R}^n\times(\sqrt{-1}\boldsymbol{R}^n\backslash\{0\}),\quad L=\sqrt{-1}S^*\boldsymbol{R}^n\cong\boldsymbol{R}^n\times\sqrt{-1}S^{n-1}.$$

We take canonical coordinates $(x,\sqrt{-1}\xi)$ of $\hat{L}$ and homogeneous canonical coordinates $(x,\sqrt{-1}\xi\infty)$ of L. Moreover, we also use inhomogeneous local coordinates (x,p) of L in a neighborhood of $\xi_n\neq 0$, where $p=(p_1,\cdots,p_{n-1})$ and $p_j=-\xi_j/\xi_n$ $(j=1,\cdots,n-1)$. The canonical map $\tau:\hat{L}\to L:(x,\sqrt{-1}\xi)\mapsto(x,\sqrt{-1}\xi\infty)$ induces a map $\tau_*:\sqrt{-1}S\hat{L}\to\sqrt{-1}SL$ (or $S\hat{L}\to SL$). Since $(\delta x,\lambda\delta\xi)\in\Gamma_{(x,\sqrt{-1}\lambda\xi)}$ for $(x,\sqrt{-1}\xi)\in\hat{L}$, $(\delta x,\delta\xi)\in\Gamma_{(x,\sqrt{-1}\xi)}$ and $\lambda>0$, we can define

$$\tilde{\Gamma}_z=\{\tau_*(x,\sqrt{-1}\xi,\sqrt{-1}v0)\in\sqrt{-1}S_zL\,;\,v\in\Gamma_{(x,\sqrt{-1}\xi)}\},$$

where $z=(x,\sqrt{-1}\xi\infty)\in L$. Lemma 2.14 in [13] gives the following

Lemma 1. *For* $z\in L$, $P_m(x,\partial/\partial x)$ *is partially micro-hyperbolic at* $(z,\pm\sqrt{-1}v0)$ *if* $(z,\sqrt{-1}v0)\in\tilde{\Gamma}_z$. *Here we refer to Kashiwara-Kawai* [6] *for the definition of partial micro-hyperbolicity* (see, also, [8]).

We write $L\hat{\times}L=\sqrt{-1}S^*\boldsymbol{R}^{2n}\backslash(\boldsymbol{R}^n\times L\cup L\times\boldsymbol{R}^n)$ and take homogeneous canonical coordinates $(x,y,\sqrt{-1}(\xi,\eta)\infty)$ of $L\hat{\times}L$. Then L is identified with the set $\{(x,x,\sqrt{-1}(\xi,-\xi)\infty)\in L\hat{\times}L\}$. If we use inhomogeneous local coordinates (x,y,p,q) of $L\hat{\times}L$ in a neighborhood of $\xi_n\neq 0$, where $p_j=-\xi_j/\xi_n$ $(j=1,\cdots,n-1)$, $q=(q_1,\cdots,q_n)$ and $q_j=-\eta_j/\xi_n$ $(j=1,\cdots,n)$, then we can also write $L=\{(x,x,p,-p,1)\in L\hat{\times}L\}$ (locally). Let Θ be a subbundle of $S_L^*(L\hat{\times}L)$ induced from the fundamental 1-form on $L\hat{\times}L$, i.e.,

$$\Theta=\{(x,x,p,-p,1,\pm(dx_n-\textstyle\sum_{j=1}^{n-1}p_jdx_j-dy_n+\sum_{j=1}^{n-1}p_jdy_j)\infty)\in S_L^*(L\hat{\times}L)\}\quad\text{(locally)}.$$

Moreover we identify S^*L with a subbundle of $S_L^*(L\hat{\times}L)$ by the map:

$$(x,p,(\textstyle\sum_{j=1}^{n}a_jdx_j+\sum_{j=1}^{n-1}b_jdp_j)\infty)$$

$$\longmapsto (x, x, p, -p, 1, (\textstyle\sum_{j=1}^n a_j dx_j - \sum_{j=1}^n a_j dy_j + \sum_{j=1}^{n-1} b_j dp_j + \sum_{j=1}^{n-1} b_j dq_j + (\sum_{j=1}^{n-1} b_j p_j) dq_n)\infty).$$

We can assume without loss of generality that $\xi_n > 0$. The canonical map H from $S_L^*(L \hat{\times} L) \backslash \Theta$ to $\sqrt{-1}SL$ is defined as follows: H maps

$$(x, x, p, -p, 1, (\textstyle\sum_{j=1}^n a_j dx_j - \sum_{j=1}^n a_j dy_j + \sum_{j=1}^{n-1} b_j dp_j + \sum_{j=1}^n b_j dq_j)\infty)$$
$$\longmapsto (x, p, \sqrt{-1}(\textstyle\sum_{j=1}^n b_j(\partial/\partial x_j) - \sum_{j=1}^{n-1}(a_j + a_n p_j)(\partial/\partial p_j))0)$$

(see [8]). Let $z \in L$ and define

$$\Gamma = H^{-1}(\tilde{\Gamma}_z) \cap S_z^* L.$$

Denote by Γ^0 the polar of Γ in $S_z L$, i.e.,

$$\Gamma^0 = \{(z, v0) \in S_z L\,;\ \langle \eta, v \rangle \leqq 0 \text{ for any } (z, \eta\infty) \in \Gamma\}.$$

Then we have

$$(1)\quad \Gamma^0 = \{(z, (\textstyle\sum_{j=1}^n a_j(\partial/\partial x_j) - \sum_{j=1}^{n-1} \xi_n^{-1}(b_j + b_n p_j)(\partial/\partial p_j))0) \in S_z L\,;$$
$$\textstyle\sum_{j=1}^n a_j \xi_j = 0 \text{ and } \sum_{j=1}^n a_j b'_j - \sum_{j=1}^n a'_j b_j \leqq 0$$
$$\text{for any } \textstyle\sum_{j=1}^n a'_j(\partial/\partial x_j) + \sum_{j=1}^n b'_j(\partial/\partial \xi_j) \in \Gamma_{(x, \sqrt{-1}\xi)}$$
$$\text{with } \textstyle\sum_{j=1}^n a'_j \xi_j = 0\} \qquad \text{for } z = (x, p) = (x, \sqrt{-1}\xi\infty) \in L.$$

Lemma 2. *Let $p(\xi)$ be a hyperbolic polynomial with respect to ϑ, and put $q(\xi') = p(\xi', 0)$, where $\xi = (\xi', \xi_n)$. Then*

$$\pi(\Gamma(p, \vartheta)^*) = \Gamma(q, \pi(\vartheta))^*,$$

where $\pi: \boldsymbol{R}^n \to \boldsymbol{R}^{n-1}: \xi \mapsto \xi'$.

Proof. Using (3.57) in [1], one can prove the lemma by the same argument as in Theorem 4.5 in [11]. Q.E.D.

Since $(\delta x, \lambda\xi + \delta\xi) \in \Gamma_{(x, \sqrt{-1}\xi)}$ for $(\delta x, \delta\xi) \in \Gamma_{(x, \sqrt{-1}\xi)}$ and $\lambda \in \boldsymbol{R}$, Lemma 2 and (1) give the following

Lemma 3. *For $z = (x, \sqrt{-1}\xi\infty) \in L$,*

$$\Gamma^0 = \{\tau_*(x, \sqrt{-1}\xi, v0) \in S_z L\,;\ -v \in \Gamma^{\sigma}_{(x, \sqrt{-1}\xi)}\}.$$

Proposition (Kashiwara-Kawai [6]). *If $Pu = f$, $z \notin$ S.S. f and $S_z G \cap \Gamma^0 = \phi$, where $G =$ S.S. u, then $z \notin$ S.S. u. Here $S_z G$ is the normal set of G along $\{z\}$* (see [6]).

Lemma 4 (Lemma 3.1 in [13]). *Let $(x, \sqrt{-1}\xi) \in \hat{L}$, and let M be a a compact set in $\Gamma_{(x, \sqrt{-1}\xi)} \subset \boldsymbol{R}^{2n}$. Then there is a neighborhood U of $(x, \sqrt{-1}\xi)$ in $\hat{L}$ such that $M \subset \Gamma_{(y, \sqrt{-1}\eta)}$ for any $(y, \sqrt{-1}\eta) \in U$.*

Let u be a solution of (CP). Then, from Proposition and Lemma 4 it easily follows that for a compact set M in $\Gamma_{(x, \sqrt{-1}\xi)}$ there is $\delta > 0$ such that S.S. $u \cap \{(y, \sqrt{-1}\eta\infty)\,;\ y_1 = x_1 - t,\ (x - y, \sqrt{-1}(\xi - \eta)) \in M^{\sigma}\} \neq \phi$ if $0 \leqq t < \delta$, $z = (x, \sqrt{-1}\xi\infty) \in$ S.S. u and $z \notin$ S.S. f. Here δ depends on M and f. Applying the same argument as in [13], we have

$$\widehat{\text{S.S.}}\, u \cap K^-_{(x, \sqrt{-1}\xi)} \cap \{(y, \sqrt{-1}\eta)\,;\ y_1 = t\} \neq \phi \qquad \text{for } -\varepsilon < t \leqq x_1$$

if $(x, \sqrt{-1}\xi) \in \widehat{\text{S.S.}}\, u$ and $K^-_{(x, \sqrt{-1}\xi)} \cap \widehat{\text{S.S.}}\, f = \phi$, where ε (>0) depends on x. Since $\widehat{\text{S.S.}}\, u \subset \{x_1 \geqq 0\}$, this proves the theorem.

Finally we remark that one can obtain the same result for hyperbolic systems, applying the results in Kashiwara-Schapira [7].

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