137. A Note on the Approximate Functional Equation for $\zeta^2(s)$. II

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1. Continuing the investigation initiated in our preceding paper [1] we show here the ζ^2-analogue of the Riemann-Siegel formula, that is, an asymptotic expansion of $\zeta^2(s)$ in the critical strip: Using the notations introduced in [1] we have

Theorem. *For $0<\sigma<1$ and $t>1$,*

$$\begin{aligned}\chi(1-s)D(s,t/2\pi)&=(t/2\pi)^{-1/4}\sum_{n=1}^{\infty} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h_0(n)\\&+(t/2\pi)^{-1/2}\frac{1}{\pi\sqrt{2}}\left\{\frac{1}{6}\left(\log\frac{t}{2\pi}+2\gamma+6\right)-\left(\sigma-\frac{1}{2}\right)\left(\log\frac{t}{2\pi}+2\gamma\right)i\right\}\\&-\frac{1}{32\pi}(t/2\pi)^{-3/4}\sum_{n=1}^{\infty} d(n)n^{-3/4}\cos(2\sqrt{2\pi nt}+\pi/4)h_0(n)\\&-\frac{\pi i}{4}\left(\sigma-\frac{1}{2}\right)(t/2\pi)^{-3/4}\sum_{n=1}^{\infty} d(n)n^{1/4}\cos(2\sqrt{2\pi nt}+\pi/4)h_1(n)\\&-\frac{\pi^2}{32}(t/2\pi)^{-3/4}\sum_{n=1}^{\infty} d(n)n^{5/4}\cos(2\sqrt{2\pi nt}+\pi/4)h_2(n)+O(1/t),\end{aligned}$$

where γ is the Euler constant, and

$$h_\nu(n)=\left(\frac{2}{\pi}\right)^{\nu+1/2}\int_0^\infty \frac{\cos(\xi+(-1)^\nu\pi/4)}{(\xi+n\pi)^{\nu+1/2}}\,d\xi.$$

Remark. At the cost of much more labour we may well replace the error $O(1/t)$ by a further approximation, but it seems that the above is sufficiently sharp for most applications.

2. The proof of our theorem which will appear elsewhere is, as may be expected, rather involved. So, in order to describe the main ideas, we show here briefly how to determine the first approximation, i.e.

$$(1)\qquad \chi(1-s)D(s,t/2\pi)=(t/2\pi)^{-1/4}\sum_{n=1}^{\infty} d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)h_0(n)+O(t^{-1/2}\log t).$$

Now, invoking the higher order approximations for $\zeta(s)$ (or rather $E(s,x)$ (see [1])) which can be obtained by the Riemann-Siegel method, we see readily that the procedure developed in [1] yields, in fact,

$$(2)\qquad \chi(1-s)\sum_{m\leq(t/2\pi)^{1/2}} m^{-s}E(s,t/2\pi m)=-\frac{1}{2}\sum_{m\leq(t/2\pi)^{1/2}}\frac{1}{m}$$

$$+\frac{1}{2\pi\varepsilon}\sum_{m\leqq(t/2\pi)^{1/2}}\frac{1}{m}\int_0^\infty e^{-(t\lambda^2/8\pi^2m^2)}\frac{\sin(\theta(t/2\pi m)\lambda/\varepsilon)}{\sin(\lambda/2\varepsilon)}d\lambda$$
$$+O(t^{-1/2}\log t),$$

where $\varepsilon=e^{\pi i/4}$ and $\theta(x)=x-[x]-1/2$. We denote this integral by I_m. Then we have, by Mellin's formula,

$$I_m=\frac{1}{2\pi i}\int_0^\infty\frac{\sin(\theta(t/2\pi m)\lambda/\varepsilon)}{\sin(\lambda/2\varepsilon)}d\lambda\int_{(Re(w)=1/4)}\Gamma(w)\left(\frac{t\lambda^2}{8\pi^2m^2}\right)^{-w}dw.$$

For a while let us assume that $t/2\pi m$ is not an integer, so that $|\theta(t/2\pi m)|<1/2$. Because of the absolute convergence of the double integral, we may change the order of integrations. Then we replace the line of the λ-integration by the contour C which starts at infinity on the positive real axis, encircles the origin once in the negative direction, excluding poles, and returns to the infinity. Further we shift the line of the w-integration to $Re(w)=5/4$. In this way we get

$$I_m=2\pi m\theta(t/2\pi m)(t/2\pi)^{-1/2}+\frac{1}{2\pi i}\int_{(Re(w)=5/4)}\frac{\Gamma(w)}{1-\exp(-4\pi i w)}$$
$$\cdot\left(\frac{t}{8\pi^2m^2}\right)^{-w}dw\int_C\lambda^{-2w}\frac{\sin(\theta(t/2\pi m)\lambda/\varepsilon)}{\sin(\lambda/2\varepsilon)}d\lambda.$$

Next, we expand C appropriately, and find

$$I_m=2\pi m\theta(t/2\pi m)(t/2\pi)^{-1/2}$$
$$+\varepsilon\int_{(Re(w)=5/4)}\frac{e^{\pi i w/2}}{\cos(\pi w)}\Gamma(w)(t/2m^2)^{-w}\sum_{n=1}^\infty n^{-2w}\sin(tn/m)dw.$$

At this stage we may drop the restriction on $t/2\pi m$ introduced above. Inserting this into (2) we get

$$\chi(1-s)\sum_{m\leqq(t/2\pi)^{1/2}}m^{-s}E(s,t/2\pi m)$$
$$(3)\qquad=-\frac{1}{2}\sum_{m\leqq(t/2\pi)^{1/2}}\frac{1}{m}+\frac{1}{\varepsilon}(t/2\pi)^{-1/2}\sum_{m\leqq(t/2\pi)^{1/2}}\theta(t/2\pi m)$$
$$+\frac{1}{2\pi}\int_{(Re(w)=5/4)}\frac{e^{\pi i w/2}}{\cos(\pi w)}\Gamma(w)(t/2)^{-w}Y(w,t)dw+O(t^{-1/2}\log t),$$

where

$$Y(w,t)=\sum_{n=1}^\infty n^{-2w}\sum_{m\leqq(t/2\pi)^{1/2}}m^{2w-1}\sin(tn/m)$$
$$=\sum_{n=1}^\infty n^{-2w}S_n(w,t),$$

say. Thus our problem is now reduced to the asymptotic evaluation of each $S_n(w,t)$.

Now, by virtue of the condition $Re(w)=5/4$, we have, by Poisson's summation formula,

$$\left(\frac{t}{2\pi}\right)^{-w}S_n(w,t)=\int_0^1 x^{2w-1}\sin(n\sqrt{2\pi t}/x)dx$$
$$+2\sum_{j=1}^\infty\int_0^1 x^{2w-1}\cos(j\sqrt{2\pi t}\,x)\sin(n\sqrt{2\pi t}/x)dx.$$

The first integral can be estimated easily. To the other integrals we apply the saddle point method with a special care for the end point $x=1$, and we see that, using the convention $0/0=0$,

$$\int_0^1 x^{2w-1}\cos(j\sqrt{2\pi t}\,x)\sin(n\sqrt{2\pi t}/x)dx$$
$$=\frac{\sqrt{\pi}}{4}(1+\operatorname{sgn}(j-n))\left(\frac{n}{j}\right)^w\frac{\sin(2\sqrt{2\pi tjn}+\pi/4)}{(2\pi tjn)^{1/4}}$$
$$+\frac{\cos((n-j)\sqrt{2\pi t})}{2(n+j)\sqrt{2\pi t}}-|\operatorname{sgn}(j-n)|\frac{\cos((j+n)\sqrt{2\pi t})}{2(n-j)\sqrt{2\pi t}}$$
$$-w(1-|\operatorname{sgn}(j-n)|)\frac{\cos(2n\sqrt{2\pi t})}{2n\sqrt{2\pi t}}$$
$$+O\{e^{\pi|w|/2}(|(n/j)^w|(tjn)^{-3/4}+|\operatorname{sgn}(j-n)|n(n-j)^{-2}t^{-1})\},$$

whence

$$(4)\qquad Y(w,t)=\frac{1}{4\sqrt{2}}\left(\frac{t}{2\pi}\right)^{w-1/4}\sum_{n=1}^{\infty}d(n)n^{-(w+1/4)}\sin(2\sqrt{2\pi tn}+\pi/4)$$
$$+O(e^{\pi|w|/2}|t^{w-1/2}|).$$

Hence, on noting that

$$\sum_{n\leqq\sqrt{x}}\theta(x/n)=-\frac{1}{2}\Delta(x)-\theta^2(x)+\frac{1}{24}+O(x^{-1/2})$$

where

$$\Delta(x)=\sum_{n\leqq x}d(n)-x(\log x+2\gamma-1)-\frac{1}{4},$$

we get, from (3) and (4),

$$\chi(1-s)\sum_{m\leqq(t/2\pi)^{1/2}}m^{-s}E(s,t/2\pi m)$$
$$=-\frac{1}{2}\sum_{m\leqq(t/2\pi)^{1/2}}\frac{1}{m}-\frac{1}{2\varepsilon}(t/2\pi)^{-1/2}\Delta(t/2\pi)$$
$$+\frac{1}{4\pi\sqrt{2}}\left(\frac{t}{2\pi}\right)^{-1/4}\sum_{n=1}^{\infty}d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)$$
$$\cdot\int_{(Re(w)=5/4)}\frac{e^{\pi iw/2}}{\cos(\pi w)}\Gamma(w)(\pi n)^{-w}dw+O(t^{-1/2}\log t).$$

Now, in the last integral we shift the line of integration to $Re(w)=1/4$, and we note that the infinite sum arising from the residue of the pole at $w=1/2$ cancels with the part of $-(2\varepsilon)^{-1}(t/2\pi)^{-1/2}\Delta(t/2\pi)$ which comes from the first term in Voronoi's formula for $\Delta(t/2\pi)$. Hence we get

$$\chi(1-s)\sum_{m\leqq(t/2\pi)^{1/2}}m^{-s}E(s,t/2\pi m)=-\frac{1}{2}\sum_{m\leqq(t/2\pi)^{1/2}}\frac{1}{m}$$
$$+\frac{1}{4\pi\sqrt{2}}\left(\frac{t}{2\pi}\right)^{-1/4}\sum_{n=1}^{\infty}d(n)n^{-1/4}\sin(2\sqrt{2\pi tn}+\pi/4)$$
$$\cdot\int_{(Re(w)=1/4)}\frac{e^{\pi iw/2}}{\cos(\pi w)}\Gamma(w)(\pi n)^{-w}dw+O(t^{-1/2}\log t).$$

Finally we note that, for $x>0$,

$$\int_{(Re(w)=1/4)} \frac{e^{\pi i w/2}}{\cos(\pi w)} \Gamma(w) x^{-w} dw = 2\sqrt{\pi} \int_0^\infty (\xi+x)^{-1/2} e^{i(\xi+\pi/4)} d\xi$$

which can be proved by constructing a differential equation satisfied by the left side. Inserting these into (5) of [1] we conclude the proof of (1).

Reference

[1] Y. Motohashi: A note on the approximate functional equation for $\zeta^2(s)$. Proc. Japan Acad., **59**A, 392–396 (1983).