103. *Zeros, Primes and Rationals*

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§ 1. **Introduction.** The connections between the primes and the zeros of the Riemann zeta function $\zeta(s)$ have been expressed in the explicit formulae since Riemann. It is Landau who showed some arithmetical connection between them; on the Riemann Hypothesis,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{0<\gamma<T} e^{ia\gamma}=\begin{cases}-\dfrac{\log p}{2\pi p^{k/2}} & \text{if } a=k\log p\\ 0 & \text{otherwise,}\end{cases}$$

where γ runs over the positive imaginary parts of the zeros of $\zeta(s)$, p is a prime and k is an integer $\geqq 1$. Here we remark the following arithmetical connection between the zeros and the rationals which we have remarked in [3] and [4].

Theorem 1. *Let α be a positive number and b be a real number $\leqq 1$. Then on the Riemann Hypothesis,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{2\pi e\alpha<\gamma\leqq T} e^{i\gamma(\log(\gamma/2\pi e\alpha))^b}$$

$$=\begin{cases}-\dfrac{e^{i\pi/4}}{2\pi}C(\alpha) & \textit{if } b=1 \textit{ and } \alpha \textit{ is rational}\\ 0 & \textit{otherwise,}\end{cases}$$

where $C(\alpha)=\mu(k)/(\sqrt{\alpha}\varphi(k))$ with the Möbius function $\mu(k)$ and the Euler function $\varphi(k)$ when $\alpha=l/k$, l and k are integers $\geqq 1$ and $(l,k)=1$.

In fact, we have proved a theorem on $\sum_{C<\gamma\leqq T} e^{if(\gamma)}$ for more general f without assuming any unproved hypothesis and given a different proof to the author's previous result (cf. [2]) which states that $f(\gamma)$ is uniformly distributed mod one, where $f(\gamma)$ may be, for example, $\gamma\log\gamma/\log\log\log\gamma$, $\gamma(\log\gamma)^b$ with $b<1$ and γ. Landau's theorem and Theorem 1 can be extended to Dirichlet L-functions $L(s,\chi)$ and these have also q-analogues (cf. [4]). We state here only a q-analogue of Theorem 1. Let $\sum_\chi'$ denote the summation over all non-principal characters $\chi \bmod q$. We suppose, for simplicity, that q runs over the primes. Let $\gamma(\chi)$ denote an imaginary part of the non-trivial zeros of $L(s,\chi)$. Then our q-analogue of Theorem 1 can be stated as follows.

Theorem 2. *Let η be an integer, α be a positive number and b be a real number $\leqq 1$. We assume the generalized Riemann Hypothesis and suppose that $T=T(q)$ satisfies $q^\nu(\log q)^B\ll T\ll q^A$, where ν is a constant depending on η, $B>B_0$ and A is an arbitrarily large constant.*

If $b=1$, $\eta=1$ *and* α *is rational and* $=l/k$, $(l,k)=1$, $1\leqq l,k$, *then for any* r *relatively prime to* k,

$$\lim_{\substack{q\to\infty \\ q\equiv r \pmod k}} \frac{1}{T\sqrt{q}} \sum_{\chi}{}' \sum_{(2\pi e\alpha/q^{\eta})<\gamma(\chi)\leqq T} e^{i\gamma(\chi)(\log(\gamma(\chi)q^{\eta}/2\pi e\alpha))^{b}} = -\frac{e^{i\pi/4}}{2\pi} C(\alpha) e^{-2\pi i l\bar{r}/k},$$

where $r\bar{r}\equiv -1 \pmod k$ *and* $C(\alpha)$ *has the same meaning as in Theorem* 1. *Otherwise,*

$$\lim_{q\to\infty} \frac{1}{T\sqrt{q}} \sum_{\chi} \sum_{(2\pi e\alpha/q^{\eta})<\gamma(\chi)\leqq T} e^{i\gamma(\chi)(\log(\gamma(\chi)q^{\eta}/2\pi e\alpha))^{b}} = 0.$$

We have proved our theorems with the remainder terms and ν in Theorem 2 may be taken as Max $(5\eta+3,\ 4\eta+20,\ 2-\eta,\ (3\eta/2)+15)$ (cf. [3] and [4] for full details).

We remark next that a slight modification of the author's [1] gives the following arithmetical connection between the zeros and the primes.

Theorem 3. *For any* $b>b_0$ *and any relatively prime integers* a *and* $k\geqq 1$, *there exists infinitely many primes which are congruent to* a mod k *and are of the form* $[\gamma\log\gamma/b\log\log\gamma]$.

We remark that $b\log\log\gamma$ in Theorem 3 can be replaced by $\Phi(\gamma)$ if $\Phi(x)$ satisfies the following conditions. $\Phi(x)$ is a positive increasing function with continuous derivatives up to three times, satisfies $\log\log x \ll \Phi(x) \ll \log x$ and $\Phi(x^c)\cong\Phi(x)$ for any positive constant c and satisfies either

1) $\Phi^{(j)}(x)/\Phi(x)=o(x^{-j}(\log x)^{-1})$ for $j=1,2,3$, or
2) $\Phi^{(j)}(x)/\Phi(x)=a_j x^{-j}(\log x)^{-1}+x^{-j}(\log x)^{-1}(u(x))^{-1}(b_j+o(1))$ for $j=1,2,3$,

where $u(x)$ is some positive increasing function which tends to ∞ as $x\to\infty$, is $\ll(\log x)^{D}$ for some positive constant D and satisfies $u(x^c)\cong u(x)$ for any positive constant c, $a_1=-a_2=a_3/2\neq 0$ and if $a_1=1$, then we suppose further that $2b_1+b_2\neq 0$, $3b_2+b_3\neq 0$ and $4b_1+5b_2+b_3\neq 0$. We remark also that if we assume the Riemann Hypothesis, $\Phi(x)$ need not be $\gg\log\log x$ but must be $\gg 1$ as in [1]. And that if $\Phi(x)\gg\log x/\log\log\log x$, then by Littlewood's theorem (cf. Theorem 9.12 of [12]) every sufficiently large integer can be written as $[\gamma\log\gamma/\Phi(\gamma)]$.

We shall prove our Theorem 3 in §2.

§2. Proof of Theorem 3. The same analysis as [1] proves Theorem 3. So we remark only how to modify it. We suppose that $X>X_0$ and $1\leqq k\ll(\log X)^{E}$ with some positive constant E. We put $f(x)=x\log x/\Phi(x)$ and $h(x)=f^{-1}(x)$ for $X>X_0$, where $\Phi(x)$ satisfies the conditions in the introduction. Since $[f(\gamma)]=p$ if and only if $p\leqq f(\gamma)<p+1$, we shall estimate

$$S\equiv\sum_{p}|\{\gamma: h(p)\leqq\gamma<h(p+1)\}|,$$

where p runs over the primes which are in $X/2<p<X$ and $\equiv a \pmod k$.

$$S=\frac{1}{\varphi(k)}\int_{X/2}^{X}\frac{L(h(u+1))-L(h(u))}{\log u}du+O(Xe^{-C\sqrt{\log X}})$$
$$+\sum\nolimits_p (S(h(p+1))-S(h(p))).$$

where $L(t)=(t/2\pi)\log t-((1+\log 2\pi)/2\pi)t$, $S(t)=(1/\pi)\arg\zeta(1/2+it)$ as usual, C is some positive constant and we have used the Riemann-von Mangoldt formula and the prime number theorem. We put $y=X^{1/\Delta}$ with $\Delta=b\log\log X$, $b>b_0$ and use Selberg's explicit formula for $S(t)$ (cf. p. 125 of [10]). Then the estimation of the last sum in S is reduced to the estimates of the following type of sums.

$$S_1=\sum_p e^{ih(p)B},\qquad S_2=\sum_p\left|\sum_{r<y^3}\frac{a(r)r^{-ih(p)}}{\sqrt{r}}\right|^2,$$
$$S_3=\sum_p\left|\sum_{r<y^{3/2}}\frac{a'(r)r^{-i2h(p)}}{r}\right|^2\quad\text{and}$$
$$S_4=\sum_p(\sigma_{y,h(p)}-1/2)^2\,\xi^{\sigma_{y,h(p)}-1/2},$$

where $B\neq 0$, r runs over the primes, $a(r)\ll\log r/\log y$ for $r<y^3$, $a'(r)\ll 1$ for $r<y^{3/2}$, $1\leqq\xi\leqq y^2$, $y^3\xi^2\ll(h(X))^{1/8}$ and $\sigma_{y,t}=1/2+2\,\mathrm{Max}_\rho(\beta-1/2,\ 2/\log y)$, ρ running over the zeros $\beta+i\gamma$ of $\zeta(s)$ for which $|t-\gamma|\leqq y^{3(\beta-1/2)}/\log y$. We remark that $h''(x)\sim-\Phi^2(h(x))(h(x))^{-1}(\log h(x))^{-3}A_1$ and $h'''(x)\sim\Phi^3(h(x))(h(x))^{-2}(\log h(x))^{-4}A_2$, where $A_1=A_2=1$ if $\Phi(x)$ satisfies 1) and $A_1=1-a_1-(2b_1+b_2)(u(h(x)))^{-1}$ and $A_2=1-a_1+(3b_2+b_3)(u(h(x)))^{-1}$ if $\Phi(x)$ satisfies 2). Consequently, the analysis in pp. 118–122 of [1] gives us

$$S_1\ll X^\delta(|B|^{1/6}+|B|^{-1/2}),$$

where δ denotes some positive number <1. S_2 and S_3 can be estimated as in p. 123 of [1], and we get

$$S_2,\ S_3\ll X/\varphi(k)\log X.$$

Now we estimate S_4.

$$S_4=\frac{X}{\varphi(k)(\log X)(\log y)^2}+\sum_p{}'(\sigma_{y,h(p)}-1/2)^2\xi^{\sigma_{y,h(p)}-1/2},$$

where the dash indicates that we sum over all p's which satisfy $\sigma_{y,h(p)}-1/2>4/\log y$, $X/2<p<X$ and $p\equiv a\ (\mathrm{mod}\ k)$. The last sum is

$$\ll\sum_\rho{}''(\beta-1/2)^2\xi^{2(\beta-1/2)}\left|\left\{X/2<p<X\,;\,p\equiv a\ (\mathrm{mod}\ k),\ |h(p)-\gamma|\leqq\frac{y^{3(\beta-1/2)}}{\log y}\right\}\right|$$
$$\ll\sum_\rho{}''(\beta-1/2)^2(y^3\xi^2)^{(\beta-1/2)}(\log X)/(k\Phi(X)\log y)+\sum_\rho{}''(\beta-1/2)^2\xi^{2(\beta-1/2)}$$
$$=S_5(\log X)/(k\Phi(X)\log y)+S_6,$$

say, where the double dash indicates that we sum over all $\rho=\beta+i\gamma$ for which $\beta>1/2+2/\log y$ and $1\ll\gamma\ll h(X)$.

$$S_5=\sum_\rho{}''\left(\int_{1/2}^{1/2+2/\log y}+\int_{1/2+2/\log y}^{\infty}\right)((\log(y^3\xi^2)(\sigma-1/2)^2$$
$$+2(\sigma-1/2))(y^3\xi^2)^{(\sigma-1/2)})d\sigma$$
$$\ll(\log y)^{-2}|\{\beta+i\gamma\,;\,\beta>1/2+2/\log y,\ 1\ll\gamma\ll h(X)\}|$$

$$+h(X)\log h(X)\int_{1/2+2/\log y}^{\infty}((\log(y^3\xi^2)(\sigma-1/2)^2$$
$$+2(\sigma-1/2))h(X)^{-1/8(\sigma-1/2)})d\sigma$$
$$\ll h(X)(\log X)(\log y)^{-2}e^{-\Delta/8}$$

by Selberg's density estimate near $\sigma=1/2$ (cf. Theorem 1 of [10]). In the same way, we get the estimate of S_6 and get

$$S_4 \ll X(\varphi(k)(\log X)(\log y)^2)^{-1}+\left(\frac{\log X}{k\Phi(X)\log y}+1\right)\frac{h(X)e^{-\Delta/8}\log X}{(\log y)^2}.$$

Consequently, we get

$$\sum_p S(h(p)), \qquad \sum_p S(h(p+1)) \ll \frac{X\log\log X}{\varphi(k)\log X}.$$

Hence we get

$$S=\frac{1}{\varphi(k)}\int_{X/2}^{X}\frac{L(h(u+1))-L(h(u))}{\log u}du+O\left(\frac{X\log\log X}{\varphi(k)\log X}\right).$$

This is $\gg X\Phi(X)/(\varphi(k)\log X)$ if $\Phi(X)\gg\log\log X$. Q.E.D.

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