114. *Some Dirichlet Series with Coefficients Related to Periods of Automorphic Eigenforms**)

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§1. In this note we construct some Dirichlet series which *generalize* those found in [9, p. 311] and [11, p. 42]. Our basic procedure is to extend the ideas in [9]. Applications will be discussed in a later note.

§2. Let m be any nonnegative integer divisible by 4. Take $R=m/2$. Let q and r be relatively prime, squarefree positive integers. Suppose that:

(2.1) $$y_0^2-ry_1^2-qy_2^2+qry_3^2\neq 0 \qquad \text{for } (y_0,y_1,y_2,y_3)\in \boldsymbol{Z}^4-\{0\}.$$

Cf. [4, pp. 115–116]. Define:

$$S=\begin{pmatrix}1&0&0\\0&-q&0\\0&0&-r\end{pmatrix}\quad S_1=\begin{pmatrix}1&0&0\\0&q&0\\0&0&r\end{pmatrix}\quad S[X]=X^tSX$$

$$n(x)=\begin{pmatrix}1&x\\0&1\end{pmatrix}\quad a(w)=\begin{pmatrix}w&0\\0&w^{-1}\end{pmatrix}\quad k(\theta)=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\quad X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\mathcal{M}_z=n(x)a(\sqrt{y})\qquad \text{for } z=x+iy,\quad x\in\boldsymbol{R},\quad y>0$$

$$\mathcal{W}\left[\begin{pmatrix}a&b\\c&d\end{pmatrix}\right]=\begin{bmatrix}\dfrac{a^2+b^2+c^2+d^2}{2} & \sqrt{q}\,(ab+cd) & \sqrt{r}\left(\dfrac{a^2-b^2+c^2-d^2}{2}\right)\\ \dfrac{ac+bd}{\sqrt{q}} & ad+bc & \sqrt{r}\left(\dfrac{ac-bd}{\sqrt{q}}\right)\\ \dfrac{a^2+b^2-c^2-d^2}{2\sqrt{r}} & \sqrt{q}\left(\dfrac{ab-cd}{\sqrt{r}}\right) & \dfrac{a^2-b^2-c^2+d^2}{2}\end{bmatrix}$$

$$\text{for } \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\boldsymbol{R})$$

$$\mathcal{V}\left[\begin{pmatrix}a&b\\c&d\end{pmatrix}\right]=\text{the analogous matrix for } S^{-1}$$

$$X_*=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad X_{**}=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad E=k\left(\frac{\pi}{2}\right)\quad \mathcal{D}=\mathcal{W}(E)=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

$$j_Q(z;m)=\frac{(cz+d)^m}{|cz+d|^m}\qquad \text{for } Q=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\boldsymbol{R})\quad \text{cf. [5, p. 357].}$$

It is easily seen that $\mathcal{W}$ and $\mathcal{V}$ are *homomorphisms* from $SL(2,\boldsymbol{R})$

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into $SL(3, \boldsymbol{R})$. In addition [for $Q \in SL(2, \boldsymbol{R})$]:

(2.2) $$\mathcal{W}(Q)^t S \mathcal{W}(Q) = S;$$

(2.3) $$\begin{cases} \mathcal{W}(Q)X_* = X_* & \text{iff } Q = k(\theta) \\ \mathcal{W}(Q)X_{**} = X_{**} & \text{iff } Q = a(w) \end{cases};$$

(2.4) $$\mathcal{V}(Q) = \mathcal{D}\mathcal{W}(Q^{-1})^t \mathcal{D};$$

(2.5) $$\mathcal{M}_{Qz} = Q\mathcal{M}_z k(\alpha) \qquad \text{where } e^{i\alpha} = \frac{|cz+d|}{cz+d}.$$

Let $\mathcal{G}_{qr}$ be the group $\{T \in PSL(2, \boldsymbol{R}) : \mathcal{W}(T) \in SL(3, \boldsymbol{Z})\}$. Cf. [3, p. 501ff]. Because of (2.1), we know that $\mathcal{G}_{qr}$ is a Fuchsian group with *compact* quotient space. Cf. [1], [3, pp. 507, 518], and [4, p. 117].

Let $\mathcal{S}$ be Schwartz space on $\boldsymbol{R}^3$. Cf. [14, p. 146]. Consider functions in $\mathcal{S}$ which satisfy $f[\mathcal{W}(k_\theta)X] \equiv e^{im\theta} f(X)$ and $h[\mathcal{V}(k_\theta)X] \equiv e^{im\theta} h(X)$. Set:

$$K_f(z) = \sum_{n \in \boldsymbol{Z}^3} f[\mathcal{W}(\mathcal{M}_z^{-1})n] \quad \text{and} \quad \mathcal{K}_h(z) = \sum_{n \in \boldsymbol{Z}^3} h[\mathcal{V}(\mathcal{M}_z^{-1})n].$$

By applying (2.5), we quickly establish that

$$K_f(T_z) = K_f(z) j_T(z; m) \quad \text{and} \quad \mathcal{K}_h(T_z) = \mathcal{K}_h(z) j_T(z; m) \qquad \text{for } T \in \mathcal{G}_{qr}.$$

The Poisson summation formula shows that

(2.7) $$K_f(z) \equiv \mathcal{K}_{\tilde{f}}(z),$$

where $\tilde{f}$ means the Fourier transform of f.

§ 3. Let Γ be *any* subgroup of $\mathcal{G}_{qr}$ having finite index. Cf. [3, p. 539] with $p=1$. The equation $n_1 = \mathcal{W}(T)n_2$ ($T \in \Gamma$) induces an obvious equivalence relation on $\boldsymbol{Z}^3$. Let $\{n_0\}$ be the corresponding set of equivalence classes. Write $\Gamma_{n_0} = \{T \in \Gamma : \mathcal{W}(T)n_0 = n_0\}$.

Introduce $L_2(\Gamma \backslash H, m)$ and $C^k(\Gamma \backslash H, m)$ as in [5, pp. 358–9] and let $\phi \in C^2(\Gamma \backslash H, m)$ be any [fixed] eigenform. Take:

(3.1) $$\begin{cases} \Delta_m \phi + s(1-s)\phi = 0 \quad \text{with} \quad s \in [1-b_m, b_m] \cup [1/2 + i\boldsymbol{R}] \\ \cdots\cdots\cdots\cdots\cdots \\ \Delta_m u = y^2(u_{xx} + u_{yy}) - imyu_x \quad \text{and} \quad b_m = \max(1, R) \end{cases}.$$

Cf. [5, p. 373] and [6, p. 370]. A formal manipulation shows that

(3.2) $$\int_{\mathcal{F}} \phi(z)\overline{K_f(z)} d\mu(z) = \sum_{\{n_0\}} \int_{FR(\Gamma_{n_0})} \phi(z)\overline{f[\mathcal{W}(\mathcal{M}_z^{-1})n_0]} d\mu(z)$$

where $FR(\Gamma_{n_0})$ denotes a fundamental region for Γ_{n_0}. A similar computation is possible with $\phi(z)\overline{\mathcal{K}_h(z)}$.

We propose to consider

$$f(X) = (\sqrt{q}\, x_2 - i\sqrt{r}\, x_3)^R e^{\pi i X^t [uS + ivS_1]X}$$

$$h(x) = \left(\frac{1}{\sqrt{q}} x_2 - \frac{i}{\sqrt{r}} x_3\right)^R e^{\pi i X^t [uS^{-1} + ivS_1^{-1}]X} \qquad \text{for } \tau = u + iv \in H.$$

The corresponding functions $K_f(z)$ and $\mathcal{K}_h(z)$ will be denoted by $\theta_m(z; \tau; S)$ and $\theta_m(z; \tau; S^{-1})$. Compare [9, p. 287(1)] and [13, pp. 86, 108]. Equation (2.7) shows that:

$$(3.3)\qquad \theta_m(z\,;\tau\,;S)\equiv\frac{(-1)^{m/4}}{\sqrt{qr}}[-iE(\tau)]^{1/2}[i\overline{E(\tau)}]^{R+1}\theta_m(z\,;E\tau\,;S^{-1}).$$

Let $t=S[n_0]$. When t is *positive*, we can write n_0 in the form $A\mathcal{W}(Q)X_*$ with $Q\in SL(2,\boldsymbol{R})$ and $A\neq 0$. It follows that Γ_{n_0} is a *cyclic group* of order $W[n_0]<\infty$.**)

Take $z_0=Q(i)$ and $w=(z-z_0)/(z-\bar{z}_0)$ as in [6, p. 342]. Thus:

$$(3.4)\qquad \phi(z)=\left(\frac{1-w}{1-\overline{w}}\right)^R\sum_{n\in Z}c_nr^{|n|}(1-r^2)^sF(s+|n|+mH_n,\,s-mH_n\,;\,1+|n|\,;\,r^2)e^{in\theta}$$

where $w=re^{i\theta}$ and $H_n=(1/2)sgn(n+1/2)$. We'll denote c_{-R} by the special symbol $E[n_0]$.

When $t<0$, we write n_0 in the form $\beta\mathcal{W}(Q)X_{**}$ with $\beta>0$. A trivial analysis of $\int_{\mathcal{F}}\theta_0(z\,;i\,;S)d\mu(z)$ shows that $\Gamma_{n0}\neq I$. Cf. (3.2). It follows that $\Gamma_{n0}=[Qa(k)Q^{-1}]$ for a uniquely determined $k>1$.

Take $\psi(z)=\phi(Qz)j_Q(z\,;m)^{-1}$ and define:

$$(3.5)\qquad I(\theta)=\int_1^{k^2}\psi(re^{i\theta})\frac{dr}{r}.$$

We'll denote $I(\pi/2)$ by the special symbol $I[n_0]$. Compare [7, p. 274].

The $\mathcal{V}$-analogs of these symbols will be denoted by $E[m_0]$ and $I[m_0]$.

Let $\Psi(a\,;c\,;z)$ be the usual confluent hypergeometric function [2, p. 255]. The following theorem is (now) obtained by *careful* computation.

Theorem 1. *We have:*

$$\int_{\mathcal{F}}\phi(z)\theta_m(z\,;\tau\,;S)d\mu(z)=\delta_{m0}\int_{\mathcal{F}}\phi(z)d\mu(z)$$
$$+\sum_{\substack{\{n_0\}\\ S[n_0]>0}}(-1)^{m/4}\pi\frac{E[n_0]}{W[n_0]}\Gamma(R+1)2^{-R}t^{R/2}(2\pi vt)^{(s-R-1)/2}\times\Psi\left[\frac{s+R+1}{2}\,;s+\frac{1}{2}\,;2\pi vt\right]e^{-\pi i\bar{\tau}t}$$
$$+\sum_{\substack{\{n_0\}\\ S[n_0]<0}}\sqrt{\pi}\,I[n_0]|t|^{R/2}(2\pi v|t|)^{(s-R-1)/2}\times\Psi\left[\frac{s-R}{2}\,;s+\frac{1}{2}\,;2\pi v|t|\right]e^{\pi i\tau|t|}.$$

A similar expansion holds for $\int_{\mathcal{F}}\phi(z)\overline{\theta_m(z\,;\tau\,;S^{-1})}d\mu(z)$.

§4. It is *very* tempting to combine Theorem 1 with equation (3.3). By analyzing the (special) case $m=0$, $\phi=1$, $\tau=iv$ we quickly establish that:

$$(4.1)\qquad \sum_{\substack{\{n_0\}\\ 0<S[n_0]\leq x}}1=O(x^{3/2}),\qquad \sum_{\substack{\{n_0\}\\ -x\leq S[n_0]<0}}lnk=O(x^{3/2}).$$

**) The generator is $Qk(\pi/W)Q^{-1}$ where $W\equiv W[n_0]$.

These (crude) estimates are very useful for convergence considerations.

Return to the case of arbitrary ϕ and take $u\approx 0$. By expanding everything in powers of u and comparing the terms of degree 0 and 1, we arrive at two basic identities (which involve *only* v). We can *now* pass to the Mellin transforms as in [9, pp. 310–311]. After some careful manipulation of hypergeometric functions, we ultimately arrive at the following proposition.

Theorem 2. *Let*:

$$F_a(\xi\,;S)=\left(\frac{1}{\sqrt{8\pi}}\right)^R\sum_{\substack{\{n_0\}\\ S[n_0]>0}}\pi\frac{E[n_0]}{W[n_0]}\Gamma(R+1)(2\pi t)^{-\xi}$$

$$F_b(\xi\,;S)=\left(\frac{1}{\sqrt{2\pi}}\right)^R\sum_{\substack{\{n_0\}\\ S[n_0]<0}}\sqrt{\pi}\,I[n_0](2\pi|t|)^{-\xi}.$$

Define $F_\mu(\xi\,;S^{-1})$ *similarly. Take* $\omega=\delta_{m0}\int_{\mathcal{F}}\phi(z)d\mu(z)$. *Then*:

(i) F_a *and* F_b *are absolutely convergent for* $Re(\xi)>3/2$;

(ii) $F_a(\xi\,;S)=(2^{-3/2}\pi^{-1/2}[\det(S)]^{-1/2}/(\xi-3/2))\omega+$ *an entire function*;

(iii) $F_b(\xi\,;S)=(2^{-3/2}[\det(S)]^{-1/2}/(\xi-3/2))\omega+$ *an entire function*;

(iv) *the same equations hold when* S *is replaced by* S^{-1};

(v) *we have*

$$\begin{pmatrix}F_a(3/2-\xi\,;S)\\ F_b(3/2-\xi\,;S)\end{pmatrix}=\frac{1}{\pi\sqrt{qr}}2^{2\xi-3/2}\Gamma\left(\xi-\frac{s}{2}\right)\Gamma\left(\xi+\frac{s-1}{2}\right)$$

$$\cdot\begin{bmatrix}\cos\pi\xi & \dfrac{\pi}{\Gamma((s-R)/2)\Gamma((1-s-R)/2)}\\ \dfrac{\pi}{\Gamma((1+s+R)/2)\Gamma((2-s+R)/2)} & \sin\pi\xi\end{bmatrix}$$

$$\cdot\begin{pmatrix}F_a(\xi\,;S^{-1})\\ F_b(\xi\,;S^{-1})\end{pmatrix}.$$

§5. The following *additional facts* should also be noted.

In the first place $E[n_0]\equiv 0$ whenever $s>1$. To check this: we integrate $\phi(z)((1-\overline{w})/(1-w))^R$ and remember that

$$\phi=O(1),\qquad r^R(1-r^2)^sF(s,s+R\,;1+R\,;r^2)\sim(\text{constant})(1-r^2)^{1-s}\quad\text{for } r\to 1.$$

Cf. (3.4) and [2, p. 107(33)].

When $s=R\geqq 2$, the function ϕ factors into $y^RF(z)$, where F is a classical holomorphic R^{th} order differential on $\Gamma\backslash H$. Cf. [5, pp. 407–408]. In this case:

$$I[n_0]=(-1)^{m/4}\frac{1}{(k^{-1}-k)^{R-1}}\int_{z_1}^{Pz_1}F(z)[cz^2+(d-a)z-b]^{R-1}dz,$$

where $P=Qa(k)Q^{-1}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and z_1 is *any* point in H. Cf. [8, p. 359(73)]. Cf. also [10], [12].

References

[1] M. Eichler: Über die Einheiten der Divisionsalgebren. Math. Ann., **114**, 635–654 (1937).

[2] A. Erdélyi: Higher Transcendental Functions. vol. 1, McGraw-Hill (1953).

[3] R. Fricke and F. Klein: Vorlesungen über die Theorie der Automorphen Funktionen. vol. 1, B. G. Teubner (1897).

[4] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro: Representation Theory and Automorphic Functions. W. B. Saunders (1969).

[5] D. A. Hejhal: The Selberg Trace Formula for $PSL(2, \boldsymbol{R})$. vol. 1, Springer Lecture Notes 548 (1976).

[6] ——: The Selberg Trace Formula for $PSL(2, \boldsymbol{R})$. vol. 2 (to appear in the Springer Lecture Note series (1982)).

[7] ——: Sur certaines séries de Dirichlet associées aux géodésiques fermées d'une surface de Riemann compacte. C. R. Acad. Sci. Paris, **294**, 273–276 (1982). *Voir aussi* **294**, 509,637 (1982).

[8] ——: Monodromy groups and Poincaré series. Bull. Amer. Math. Soc., **84**, 339–376 (1978).

[9] H. Maass: Über die räumliche Verteilung der Punkte in Gittern mit indefiniter Metrik. Math. Ann., **138**, 287–315 (1959).

[10] T. Oda: On modular forms associated with indefinite quadratic forms of signature $(2, n-2)$. Math. Ann., **231**, 97–144 (1977).

[11] T. Shintani: On zeta functions associated with the vector space of quadratic forms. J. Fac. Sci. Univ. Tokyo, **22**, 25–65 (1975).

[12] ——: On the construction of holomorphic cusp forms of half-integral weight. Nagoya Math. J., **58**, 83–126 (1975).

[13] C. L. Siegel: Gesammelte Abhandlungen. vol. 3, Springer-Verlag (1966).

[14] K. Yosida: Functional Analysis. 2nd ed., Springer-Verlag (1968).