81. *Path Integral for the Dirac Equation in Two Space-Time Dimensions*

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Introduction. With his physical postulates Feynman ([4], [5]) conceived the eminent idea of path integral in quantum mechanics. Kac [6] has given a rigorous realization of Feynman's idea for pure-imaginary-time quantum mechanics. Namely, he has represented the solution of the heat equation by the Wiener measure over the Brownian path space. It is called the Feynman-Kac formula.

The aim of the present note is to give a path integral formula for the solution of the Dirac equation in two-dimensional space-time. It shows a very close analogy with the Feynman-Kac formula, but the path space measure constructed is other than the Wiener measure.

Some physical treatments of the problem are found in Feynman [5, Chap. 2, 2–4], Riazanov [8] and Rosen [9].

1. Statement of result. The Dirac equation in two space-time dimensions has the following form:

$$\frac{\partial}{\partial t}\phi(t,x)=\left[-\alpha\left(\frac{\partial}{\partial x}-iA_1(t,x)\right)-im\,\beta+iA_0(t,x)\right]\phi(t,x),\qquad t\in\boldsymbol{R},\quad x\in\boldsymbol{R}. \tag{1.1}$$

Here α and β are 2×2 Hermitian symmetric matrices with $\alpha^2=\beta^2=1$ and $\alpha\beta+\beta\alpha=0$. Both $A_0(t,x)$ and $A_1(t,x)$ are real-valued functions on $\boldsymbol{R}^2$. The constant m is the rest mass of the particle, and other physical units are chosen such that the light velocity c and the Planck constant $\hbar$ equal 1.

Now put $x_0=t$ and $x_1=x$ to rewrite (1.1) as

$$iH\phi(x)\equiv\left[\left(\frac{\partial}{\partial x_0}-iA_0(x)\right)+\alpha\left(\frac{\partial}{\partial x_1}-iA_1(x)\right)+im\,\beta\right]\phi(x)=0, \tag{1.2}$$

where $x=(x_0,x_1)\in\boldsymbol{R}^2$. Introduce the proper time s (cf. [8]) to consider the Cauchy problem for

$$\frac{\partial}{\partial s}\psi(s,x)=iH\psi(s,x),\quad s\in\boldsymbol{R},\quad x\in\boldsymbol{R}^2 \tag{1.3}$$

with initial data $\psi(0,x)=g(x)$.

Then the solution of (1.3) admits the following path integral representation. We set $A(x)=(A_0(x),A_1(x))$, and use the physicist inner product $\langle\cdot,\cdot\rangle$.

Theorem. *Let $A(x)$ be an $\boldsymbol{R}^2$-valued, C^1 function defined in $\boldsymbol{R}^2$.*

Then for every $s>0$ and for every f and $g \in \mathcal{S}(\boldsymbol{R}^2)$ there exists a countably additive complex measure $\nu_{s,f,g}$ on the Banach space $C([0,s];\boldsymbol{R}^2)$ of the continuous paths $X:[0,s]\to\boldsymbol{R}^2$ such that

$$\begin{aligned}\langle f, e^{isH}g\rangle &= \langle f(\cdot), \psi(s,\cdot)\rangle \\ (1.4)\qquad &= \int d\nu_{s,f,g}(X)\exp\left\{i\int_0^s A(X(\tau))dX(\tau)\right\}.\end{aligned}$$

The support of $\nu_{s,f,g}$ is included in the set of those Lipschitz continuous paths $X(\tau)$ with Lipschitz constant $\sqrt{2}$ which connect the points of supp g *with the points of* supp f.

Remarks. 1. A path integral representation of the solution $\psi(s,x)$ itself in terms of a 2×2 complex matrix-valued measure $\nu_{s,x}$ on $C([0,s];\boldsymbol{R}^2)$ is also possible.

2. The formula (1.4) yields the expression of the Green function for the Dirac equation (1.2) if it exists:

$$i\langle f, H^{-1}g\rangle = \lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon s}ds\int d\nu_{s,f,g}(X)\exp\left\{i\int_0^s A(X(\tau))dX(\tau)\right\}.$$

3. A further study shows that $A(x)$ is allowed to depend on s, so that a similar path integral representation holds for the solution $\phi(t,x)$ of the Cauchy problem for (1.1) with initial data $\phi(0,x)=g(x)$. The corresponding path space measure has, for $m>0$, the support on those Lipschitz continuous paths $X:[0,t]\to\boldsymbol{R}$ whose slopes are smaller than or equal to the light velocity 1. If $m=0$, the support is on the set of the paths with slopes exactly equal to the light velocity 1.

4. Daletskii ([1], [2, §§5, 8], [3]) studied some related problems, but no countably additive path space measure was constructed.

2. Sketch of proof. Our construction of the path space measure will make use of Nelson's method [7] of construction of the Wiener measure.

Let $\dot{\boldsymbol{R}}^2$ be the one-point compactification of $\boldsymbol{R}^2$, and $\boldsymbol{X}_s=\prod_{[0,s]}\dot{\boldsymbol{R}}^2$ the product of the uncountably many copies of $\dot{\boldsymbol{R}}^2$. We may regard $\boldsymbol{X}_s$ as the set of all paths $X:[0,s]\to\dot{\boldsymbol{R}}^2$, possibly discontinuous and possibly passing through infinity. Equipped with the product topology, $\boldsymbol{X}_s$ is a compact Hausdorff space by the Tychonoff theorem. Let $C(\boldsymbol{X}_s)$ be the Banach space of the continuous functions on $\boldsymbol{X}_s$, and $C_{fin}(\boldsymbol{X}_s)$ its subspace consisting of all $\Phi(X)$ for which there exist a finite partition $0=s_0<s_1<\cdots<s_n=s$ of the interval $[0,s]$, and a bounded continuous function $F(x^0,x^1,\cdots,x^n)$ on $(\dot{\boldsymbol{R}}^2)^{n+1}$ such that $\Phi(X)=F(X(s_0),X(s_1),\cdots,X(s_n))$. By the Stone-Weierstrass theorem $C_{fin}(\boldsymbol{X}_s)$ is dense in $C(\boldsymbol{X}_s)$.

Let $K(s,x)$ be the fundamental solution for the Cauchy problem for (1.3) with $A(x)\equiv 0$, which is given by

$$K(s,x)=\frac{1}{2}\delta(x_0+s)\left[\frac{\partial}{\partial s}+\alpha\frac{\partial}{\partial x_1}+im\beta\right](J_0(m(s^2-x_1^2)^{1/2})\theta(s-|x_1|)),\qquad s>0.$$

Here $J_0(t)$ is the Bessel function of order zero, and $\theta(t)$ the Heaviside function: $\theta(t)=1$ for $t>0$, $=0$ for $t<0$.

For each $s>0$ and for each f and $g \in \mathcal{S}(\boldsymbol{R}^2)$ define a linear form $L_{s,f,g}$ on $C_{fin}(X_s)$ by

$$L_{s,f,g}(\Phi)=\overbrace{\int_{R^2}\cdots\int_{R^2}}^{n+1}\overline{f(x^n)}K(s_n-s_{n-1},\, x^n-x^{n-1})\cdots K(s_2-s_1,\, x^2-x^1)$$
$$\cdot K(s_1,\, x^1-x^0)F(x^0, x^1, \cdots, x^n)g(x^0)dx^0dx^1\cdots dx^n.$$

The following lemma plays a crucial role.

Lemma. *$L_{s,f,g}$ is well-defined on $C_{fin}(X_s)$ and there exists a constant C independent of s such that, for every $\Phi \in C_{fin}(X_s)$,*

$$|L_{s,f,g}(\Phi)|\le Ce^{ms}\|\Phi\|_\infty.$$

By this lemma and by denseness of $C_{fin}(X_s)$ in $C(X_s)$, $L_{s,f,g}$ can be extended to a continuous linear form on $C(X_s)$. Thus by the Riesz theorem there exists a unique regular Borel measure $\nu_{s,f,g}$ on X_s such that, for every $\Phi \in C(X_s)$,

$$L_{s,f,g}(\Phi)=\int_{X_s} d\nu_{s,f,g}(X)\Phi(X).$$

In view of the property of the kernel $K(s, x)$ we can see that $\nu_{s,f,g}$ has the support in $C([0, s]; \boldsymbol{R}^2)$, and further in the set of the Lipschitz continuous paths with the property mentioned in Theorem.

To establish (1.4) define the operator

$$(T(r)g)(x)=\int_{R^2} K(r, x-y)e^{iA(y)(x-y)}g(y)dy$$

for $g \in \mathcal{S}(\boldsymbol{R}^2)$. Then we obtain for $f \in \mathcal{S}(\boldsymbol{R}^2)$ with $s_j=js/n$

$$\left\langle f,\, T\left(\frac{s}{n}\right)^n g\right\rangle=\int d\nu_{s,f,g}(X)\exp\left\{i\sum_{j=1}^{n} A(X(s_{j-1}))(X(s_j)-X(s_{j-1}))\right\}.$$

It is shown that as $n\to\infty$, the left-hand side converges to $\langle f, e^{isH}g\rangle$, while the right-hand side does to the last member of (1.4).

Detailed proofs and extensions of the results will appear elsewhere.

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