18. On some Schrödinger Type Equations

By Sigeru MIZOHATA
Department of Mathematics, Kyoto University

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1. Introduction. We are concerned with the following evolution equation defined for $(x, t) \in R^n \times R^1$:

$$L(u)=i\partial_t u+\sum_{j,k} \partial_j(a_{jk}(x)\partial_k u)+\sum_j b_j(x)\partial_j u+c(x)u=f(x, t), \tag{1.1}$$

with initial data $u_0(x) \in L^2(R^n)$ at $t=0$, where $\partial_j=\partial/\partial x_j$ and $a_{jk}(x)$ are real-valued and bounded with all their derivatives with $a_{jk}(x)=a_{kj}(x)$. Moreover we assume the uniform ellipticity: $\sum a_{jk}(x)\xi_j\xi_k \geqslant \delta|\xi|^2$ $(\delta>0)$.

Recently J. Takeuchi treated this type problem for more general equations ([3]). He proposes a (sufficient) condition for the equation (1.1) to be L^2-wellposed. This terminology means the following: For any initial data $u_0(x) \in L^2(R^n)$, and any continuous function $t \mapsto f(\cdot, t)$ in L^2, the equation (1.1) has a unique solution $u(\cdot, t)$, continuous in L^2, so that by Banach's closed graph theorem there exists a constant C such that for any $t \in [-1, 1]$, it holds

$$\|u(t)\| \leqslant C\left(\|u(0)\|+\left|\int_0^t \|f(s)\|ds\right|\right). \tag{1.2}$$

The purpose of this note is to investigate the neccessary condition for this problem. Let

$$a(x, \xi)=\sum a_{jk}(x)\xi_j\xi_k.$$

We consider the integral curves, called bicharacteristic strips, of the corresponding Hamilton-Jacobi equation

$$\begin{cases} \dfrac{dx}{dt}=a_\xi(x, \xi) \\ \dfrac{d\xi}{dt}=-a_x(x, \xi). \end{cases} \tag{1.3}$$

Denote its integral curve issuing from (x_0, ξ^0) at $t=0$ by $(x(x_0, t, \xi^0), \xi(x_0, t, \xi^0))$. Then the condition that we propose is:

(B) $\mathrm{Re}\int_0^t \sum_j b_j(x(x_0, s, \xi^0))\xi_j(x_0, s, \xi^0)ds$ remain bounded for all $(x_0, \xi^0) \in R^n \times R^n \setminus \{0\}$ and $t \in R^1$.

We assume here the following:

(A) For any $(\xi^0, t) \in R^n \setminus \{0\} \times R^1$ fixed, the mapping $x_0 \mapsto x(x_0, t, \xi^0)$ is a diffeomorphism from R^n onto R^n.

Let us note that, under this assumption, there exists the global phase function $S(x, t, \xi^0)$, which is defined as the solution of Hamilton-Jacobi's equation

(1.4) $$S_t+a(x, S_x)=0,$$
with $S(x, 0, \xi^0)=x\xi^0$. Then we have

Theorem. *Under the assumption* (A), *the condition* (B) *is necessary for the* L^2*-wellposedness of* (1.1).

Remark. We assume $b_j(x)\in C^1$ and $c(x)$ is locally bounded. If we are concerned only with forward Cauchy problem, the condition (B) should be replaced by

$(B)_+$ $\mathrm{Re}\int_0^t b(x(x_0, s, \xi^0))\xi(x_0, s, \xi^0)ds$ remain bounded from below for all $(x_0, \xi^0)\in R^n\times R^n\setminus\{0\}$ and $t\geqslant 0$.

For the proof, we use asymptotic solutions which are fairly familiar (see [1], [2]). In other words, we look at wave packets moving along classical trajectories corresponding to Hamiltonian $a(x, \xi)$.

2. Approximate solutions. We show the necessity of $(B)_+$. Let us suppose $(B)_+$ is violated. Then there exist x_0^0 $(=(x_{0,1}, \cdots, x_{0,n}))$ and $\xi^0\in R^n\setminus\{0\}$ and $t_1>0$, such that

(2.1) $$\mathrm{Re}\int_0^{t_1} b(x(x_0^0, s, \xi^0))\xi(x_0^0, s, \xi^0)ds\leqslant -\log(2C),$$

where C is the constant in (1.2). We seek an approximate solution of the form

$$u(x, t)=e^{iS(x,t,\xi^0)}v(x, t; \xi^0).$$

$$e^{-iS}[i\partial_t+\sum \partial_j a_{jk}(x)\partial_k+\sum b_j(x)\partial_j](e^{iS}v)$$
$$=[-S_t+i\partial_t+\sum(\partial_j+iS_{x_j})a_{jk}(x)(\partial_k+iS_{x_k})+\sum b_j(iS_{x_j}+\partial_j)]v(x, t, \xi^0).$$

Taking account of (1.4), [...] becomes

$$i[\partial_t+2\sum_{j,k} a_{jk}(x)S_{x_j}\partial_k+\sum \partial_j(\sum_k a_{jk}(x)S_{x_k})+\sum b_j(x)S_{x_j}]v$$
$$+\sum_{j,k}\partial_j(a_{jk}(x)\partial_k v)+\sum b_j(x)\partial_j v.$$

We define v as a solution of the transport equation

(E) $$\{\partial_t+\sum_j a_{\xi_j}(x, S_x)\partial_j\}v+\{\sum_j \partial_j(\sum_k a_{jk}(x)S_{x_k})+\sum_j b_j(x)S_{x_j}\}v=0.$$

Let $\varphi(x_0)$ be a smooth function with small compact support around $x_0=x_0^0$. Let the inverse mapping of $x_0\mapsto x(x_0, t, \xi^0)$ be $x_0(x, t, \xi^0)$ then the integration of the relation (E) along the bicharacteristic curve Γ through the point (x, t) gives

$$v(x, t; \xi^0)=\exp\Big[-\sum_j\int_0^t b_j(x(x_0, s, \xi^0))\xi_j(x_0, s, \xi^0)ds\Big]$$
$$\times\exp\Big[-\int_0^t\sum_j \partial_j(\sum_k a_{jk}(x)S_{x_k})|_\Gamma ds\Big]\varphi(x_0(x, t, \xi^0)).$$

Now it is easy to see that the second factor of the right-hand side is equal to $|dx/dx_0|_{t=t}^{-1/2}$ where dx/dx_0 is Jacobian. Hence the above expression is

(2.2) $$v(x, t; \xi^0)=\exp\Big[-\int_0^t\sum_j b_j(x)\xi_j|_\Gamma ds\Big]\left|\frac{dx}{dx_0}\right|_{t=t}^{-1/2}\varphi(x_0(x, t, \xi^0)).$$

3. Proof of Theorem. Up to now we fixed ξ^0. Now we replace

ξ^0 by $\lambda\xi^0$, λ being positive parameter tending to ∞. Denote the corresponding solution by $v_\lambda(x, t)$. To see clearly the dependence of $v_\lambda(x, t)$ on λ, first we observe

$$\begin{cases} x(x_0, t, \lambda\xi^0) = x(x_0, \lambda t, \xi^0) \\ \xi(x_0, t, \lambda\xi^0) = \lambda\xi(x_0, \lambda t, \xi^0). \end{cases} \tag{3.1}$$

In fact, the function $t \mapsto (x(x_0, \lambda t, \xi^0), \lambda\xi(x_0, \lambda t, \xi^0))$ satisfies Hamilton-Jacobi's equation (1.3) with initial data $(x_0, \lambda\xi^0)$ at $t=0$. Then by uniqueness, (3.1) follows. This shows in particular the projection on the x-space of the bicharacteristic strip is unchanged. Next, since in the expression of (2.2) (in view of (3.1)), we have

$$\int_0^t b(x(x_0, s, \lambda\xi^0))\xi(x_0, s, \lambda\xi^0)ds = \int_0^{\lambda t} b(x(x_0, s, \xi^0))\xi(x_0, s, \xi^0)ds.$$

Put

$$w(x, t, \xi^0) = \left|\frac{dx}{dx_0}\right|^{-1/2} \varphi(x_0(x, t, \xi^0)).$$

Recalling the relation $x_0(x, t, \lambda\xi^0) = x_0(x, \lambda t, \xi^0)$, we see easily that $w(x, t, \lambda\xi^0) = w(x, \lambda t, \xi^0)$. Thus,

$$v_\lambda(x, t) \equiv v(x, t; \lambda\xi^0) = v(x, \lambda t; \xi^0). \tag{3.2}$$

Another important property is that the mapping $\varphi(x) \mapsto w(\cdot, t, \xi^0)$ is unitary, namely L^2-norm is preserved. Now v_λ satisfies

$$L(e^{iS_\lambda}v_\lambda) = f_\lambda(x, t), \tag{3.3}$$

$$\begin{cases} S_\lambda(x, t) = S(x, t, \lambda\xi^0) \\ f_\lambda(x, t) = ie^{iS_\lambda}[\sum_{j,k} \partial_j \cdot a_{jk}(x)\partial_k + \sum_j b_j(x)\partial_j + ic(x)]v_\lambda(x, t). \end{cases} \tag{3.4}$$

An important property of $f_\lambda(x, t)$ is that, in view of its form, we have $|f_\lambda(x, t)| = |f_1(x, \lambda t)|$, f_1 being defined by (3.4) replacing there v_λ by $v(x, t, \xi^0)$. Thus

$$\int_0^t \|f_\lambda(\cdot, s)\| ds = \int_0^t \|f_1(\cdot, \lambda s)\| ds = \frac{1}{\lambda}\int_0^{\lambda t} \|f_1(\cdot, s)\| ds. \tag{3.5}$$

Now we apply (1.2) to the relation (3.3). Since S_λ is real, we have

$$\|v_\lambda(\cdot, t)\| \leqslant C\left(\|\varphi\| + \int_0^t \|f_\lambda(\cdot, s)\| ds\right). \tag{3.6}$$

First we put here $t = t_1/\lambda$. Then by (3.5), the integral term is $\frac{1}{\lambda}\int_0^{t_1} \|f_1(\cdot, s)\| ds$. Next, remark that the support of $w(x, t_1, \xi^0)$ is concentrated around $x^{(1)} = x(x_0^0, t_1, \xi^0)$, moreover its diameter can be made as small as we desire by shrinking the support of φ to x_0^0. Thus by (2.1) we can assume, in the expression (2.2) replaced t by t_1, exponential term is greater than $3C/2$ in absolute value. Thus, for $t = t_1/\lambda$, we have

$$\|v_\lambda(\cdot, t)\| \geqslant \frac{3C}{2}\|\varphi\|.$$

Thus (3.6) implies the following inequality

$$\frac{3C}{2}\|\varphi\| \leqslant C\left(\|\varphi\| + \frac{1}{\lambda}\int_0^{t_1} \|f_1(\cdot, s)\| ds\right),$$

which is impossible when $\lambda \to \infty$. Thus we proved Theorem.

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