116. *Differentiability of Riemann's Function*

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1. Introduction. In this paper we discuss on the differentiability of the function

$$f(x)=\sum_{n=1}^{\infty} \sin n^2x/n^2.$$

Riemann proposed the problem that the function is nowhere differentiable, [2] and [8]. About the problem, J. P. Kahane [3] has investigated lacunary series. It was solved by J. Gerver [4] [5]. First G. H. Hardy [6] proved that the function is not differentiable at the point $\xi\pi$ where ξ is irrational or is a rational of the form $(2A+1)/2B$ or $2A/(4B+1)$. Later Gerver proved that $f(x)$ is differentiable at all points $(2A+1)\pi/(2B+1)$ with derivative $-1/2$, and not differentiable at the points $2A\pi/(2B+1)$.

The purpose of this paper is to give a shorter proof of the differentiability as well as a finer estimate of the function at points of rational multiple of π.

We states the following

Theorem 1. *The function*

$$F(x)=\sum_{n=1}^{\infty} \exp(in^2\pi x)/n^2\pi i$$

have the following behavior near $x=q/p$, where p is a positive integer and q is an integer such that q/p is an irreducible fraction,

$$\begin{aligned}(1)\qquad & F(x+h)-F(x)\\ & =R(p,q)p^{-1/2}\exp\left(\frac{\pi i}{4}\operatorname{sgn} h\right)|h|^{1/2}\operatorname{sgn} h-\frac{h}{2}+O(|h|^{3/2})\end{aligned}$$

as $h\to 0$ where $\operatorname{sgn} h=h/|h|$ if $h\neq 0$, $\operatorname{sgn} h=0$ if $h=0$, and $R(p,q)$ is a constant defined by

$$(2)\qquad R(p,q)=\begin{cases}\left(\dfrac{q}{p}\right)\exp\left(\dfrac{-\pi i}{4}(p-1)\right) & \text{if } p \text{ is odd and } q \text{ even},\\ \left(\dfrac{p}{|q|}\right)\exp\left(\dfrac{\pi i}{4}q\right) & \text{if } p \text{ is even and } q \text{ odd},\\ 0 & \text{if } p \text{ and } q \text{ are odd},\end{cases}$$

with the Jacobi's symbol (p/q) (see [7]).

This theorem easily shows Gerver's results and gives a finer estimate of $f(x)$ at the points of rational multiple of π. The author wishes to thank Prof. Jean Pierre Kahane for helpful suggestions.

2. Proofs. In order to prove the theorem we prepare an equation of F

$$F(x)-F(0)=\exp\left(\frac{\pi i}{4}\right)\sqrt{x}-\frac{x}{2}+\exp\left(\frac{\pi i}{4}\right)x^{3/2}F\left(-\frac{1}{x}\right) \tag{3}$$
$$-\frac{3}{2}\exp\left(\frac{\pi i}{4}\right)\int_0^x u^{1/2}F\left(-\frac{1}{u}\right)du \qquad x>0.$$

To obtain (3), if we set $g(x)=\sum_{n=-\infty}^{\infty}\exp(-n^2\pi x)$, then we have the *theta-relation* ([1])

$$g(x)=x^{-1/2}\sum_{n=-\infty}^{\infty}\exp(-n^2\pi/x) \qquad x>0.$$

First notice that this relation holds, by analytic continuation, also for complex x with $\operatorname{Re} x>0$. We have, for $u, y>0$

$$g(y-iu)=\exp\left(\frac{\pi i}{4}\right)(1/\sqrt{u+iy})\sum_{n=-\infty}^{\infty}\exp(-n^2\pi i/(u+iy)). \tag{4}$$

Integrating (4) by u and simple integration by part shows the following equation.

$$\int_0^x g(y-iu)du$$
$$=\exp\left(\frac{\pi i}{4}\right)\left[2\sqrt{u+iy}+(u+iy)^{3/2}\sum_{n\neq 0}\frac{1}{n^2\pi i}\exp(-n^2\pi i/(u+iy))\right]_0^x$$
$$-\exp\left(\frac{\pi i}{4}\right)\int_0^x\frac{3}{2}(u+iy)^{1/2}\sum_{n\neq 0}\frac{1}{n^2\pi i}\exp(-n^2\pi i/(u+iy))du.$$

Because

$$\int_0^x g(y-iu)du=x+\sum_{n\neq 0}\exp(-n^2\pi y)(\exp(in^2\pi x)-1)/n^2\pi i,$$

we can obtain the equation (3) by letting y tend to 0.

From (3) we have that as $h\to 0$ the function F satisfies

$$F(h)-F(0)=\exp\left(\frac{\pi i}{4}\operatorname{sgn} h\right)\sqrt{|h|}\operatorname{sgn} h-\frac{h}{2}+O(|h|^{3/2}), \tag{5}$$

$$F(h+1)-F(1)=-\frac{h}{2}+O(|h|^{3/2}), \tag{6}$$

because $F(x+1)=2^{-1}F(4x)-F(x)$.

Let us prove the following

Lemma 2. *For fixed $x\neq 0$ assume there exist constants $c(1)$ and $c(-1)$ such that*

$$F(x+h)-F(x)=c(\operatorname{sgn} h)\sqrt{|h|}\operatorname{sgn} h-2^{-1}h+O(|h|^{3/2})$$

as $h\to 0$, Then for $y=-1/x$

$$F(y+h)-F(y)=y^{1/2}\exp\left(\frac{\pi i}{4}\operatorname{sgn} y\right)c(\operatorname{sgn} h)\sqrt{|h|}\operatorname{sgn} h \tag{7}$$
$$-2^{-1}h+O(|h|^{3/2}) \qquad \textit{as } h\to 0.$$

Proof. By assumption we have

$$F\left(-\frac{1}{y+h}\right)-F\left(-\frac{1}{y}\right)=c(\operatorname{sgn} h)\frac{1}{|y|}\sqrt{|h|}\operatorname{sgn} h-\frac{1}{2y^2}h+O(|h|^{3/2}).$$

So if $y>0$, by (3) we can obtain the relation (7) for $y>0$. If $y<0$, by taking complex conjugate of (3) the same discussion can be applied. Therefore the lemma is proved.

We also prepare the properties of R by means of Jacobi's symbol (see [7]). For positive integer p and integer q which is relatively prime with p, we have

(8) $\quad R(p,q)=R(p,q')\quad$ if q is congruent to q' modulo $2p$.

For odd p the truth of (8) is obvious. Suppose p is even and $p=2^k\cdot a$ where k is positive and a is odd. Using the properties of the Jacobi's symbol we can show

$$R(p,q)=(-1)^{k8^{-1}(q^2-1)+(a-1)4^{-1}(q-1)\operatorname{sgn} q}\exp\left(\frac{\pi i}{4}q\right)\left(\frac{q}{a}\right).$$

Because q' is congruent to q modulo $2^{k+1}\cdot a$, it follows that

$$\frac{R(p,q')}{R(p,q)}=(-1)^{(q'-q)4^{-1}(kq+1)}.$$

If 8 divides $q'-q$ then the above term is equal to 1. If 8 does not divide $q'-q$ then $k=1$ and $kq+1$ is even, and hence the term is equal to 1. Thus the truth of (8) follows.

Proof of Theorem 1. Put $x=q/p$ where p is a positive integer and q/p an irreducible fraction, we shall prove (1) by induction in $2p+|q|$. If $2p+|q|\leqq 3$, the truth of (1) follows from (5) and (6). Let $2p+|q|>3$. Suppose (1) is true for all (p',q') with $2p'+|q'|<2p+|q|$.

1) If $|q|\geqq p$, by $2p+|q|>3$, we can choose an integer q' such that $|q'|<p$ and q' is congruent to q modulo $2p$. We have

$$F\left(\frac{q}{p}+h\right)-F\left(\frac{q}{p}\right)=F\left(\frac{q'}{p}+h\right)-F\left(\frac{q'}{p}\right).$$

Thus by using assumption of induction for (p,q'), (1) follows from the relation (8).

2) If $|q|<p$, then by putting $p'=|q|$, $q'=-p\operatorname{sgn} q$, we get $q'/p'=-p/q$. By assumption of induction we get

$$F\left(-\frac{p}{q}+h\right)-F\left(-\frac{p}{q}\right)$$
$$=R(p',q')p'^{-1/2}\exp\left(\frac{\pi i}{4}\operatorname{sgn} h\right)\sqrt{|h|}\operatorname{sgn} h-\frac{h}{2}+O(|h|^{3/2})$$

as $h\to 0$. In order to prove Theorem 1, by Lemma 2, it is sufficient to show that

$$(9)\qquad R(p',q')\exp\left(\frac{\pi i}{4}\operatorname{sgn} q\right)=R(p,q).$$

First if p and q are odd, then $R(p',q')=R(p,q)$. Thus (9) follows. Second if p is odd and q even, then (9) follows from the relation

$$\left(\frac{|q|}{p}\right)=(-1)^{(p-1)(\operatorname{sgn} q-1)/4}\left(\frac{q}{p}\right).$$

Lastly if p is even and q odd, then (9) follows from the same argument in the second case.

References

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