8. *On Conformal Diffeomorphisms between Product Riemannian Manifolds*

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(Communicated by Kunihiko KODAIRA, M. J. A., Jan. 12, 1981)

In this note, a conformal diffeomorphism means a non-homothetic conformal one. Let $M=M_1\times M_2$ and $M^*=M_1^*\times M_2^*$ be connected product Riemannian manifolds of dimension $n\geqq 3$, and denote the metric product structures by (M, g, F) and (M^*, g^*, G) respectively. Several geometers [1]–[3], [5]–[8] proved non-existence of global conformal diffeomorphism between complete product Riemannian manifolds with certain properties. The purpose of this note is to announce the following

Theorem. *If M and M^* are complete product Riemannian manifolds, then there is no global conformal diffeomorphism of M onto M^* such that it does not commute the product structures F and G, $FG\neq GF$, somewhere in M.*

This is an improvement of the main theorem in a previous paper [5]. As the contraposition, a conformal diffeomorphism of M onto M^* has to commute the product structures F and G everywhere in M, and an example of such a conformal diffeomorphism was given in [5].

Outline of the proof. Let M_1 and M_2 be of dimension n_1 and n_2 respectively, $n_1+n_2=n$, and (x^i, x^p) a separate coordinate system of M, (x^i) belonging to M_1 and (x^p) to M_2. Latin indices run on

$$i, j, k=1, 2, \cdots, n_1;\ p, q, r=n_1+1, \cdots, n$$

respectively, and Greek indices κ, λ, μ, ν on the range 1 to n. The metric tensor $g=(g_{\mu\lambda})$ of M has pure components g_{ji} and g_{qp} only with respect to the separate coordinate system (x^i, x^p).

A conformal diffeomorphism f of M to M^* is characterized by a change of the metric tensors

$$g^*_{\mu\lambda}=\frac{1}{\rho^2}g_{\mu\lambda}, \tag{1}$$

ρ being a positive-valued scalar field. The integrability of the product structure G with respect to g^* in M^* is equivalent to

$$\nabla_\mu G_{\lambda\kappa}=-\frac{1}{\rho}(G_{\mu\lambda}\rho_\kappa+G_{\mu\kappa}\rho_\lambda-g_{\mu\lambda}G_{\kappa\nu}\rho^\nu-g_{\mu\kappa}G_{\lambda\nu}\rho^\nu), \tag{2}$$

where ∇ indicates covariant differentiation in M and $\rho_\lambda=\nabla_\lambda\rho$, $\rho^\nu=\rho_\lambda g^{\lambda\nu}$. Denote the gradient vector field (ρ^λ) by Y, the parts (ρ^i) along to M_1 by Y_1 and (ρ^p) to M_2 by Y_2. Put $\Phi=|Y|^2=\rho_\lambda\rho^\lambda=|Y_1|^2+|Y_2|^2$ and

$$N_1=\{P|Y_1(P)=0\}, \qquad N_2=\{P|Y_2(P)=0\},$$
$$U=\{P|Y_1(P)\neq 0,\ Y_2(P)\neq 0\}, \qquad V=\{P|FG\neq GF \text{ at } P\}.$$

Starting from the equation (2), we have the inclusion relations

$$U\subset V\subset M-N_1\cap N_2.$$

By definition, a *special concircular* scalar field ρ satisfies

$$(3) \qquad \nabla_\mu\rho_\lambda=(k\rho+b)g_{\mu\lambda},$$

k and b being constants. The trajectories of $Y=(\rho^\lambda)$ are geodesics, called *ρ-curves*. In a neighborhood of an ordinary point P of ρ, $Y(P)\neq 0$, there is an adapted coordinate system (u, u^α), $\alpha, \beta, \gamma=2, \cdots, n$, such that u is the arc-length of ρ-curves, ρ is a function of u only, and the metric of M is given in the form

$$ds^2=du^2+\{\rho'(u)\}^2\overline{ds^2},$$

where $\overline{ds^2}=f_{\gamma\beta}du^\gamma du^\beta$ is the metric form of an $(n-1)$-dimensional Riemannian manifold $\overline{M}$. If M is complete and $k<0$, then M is a sphere and $\overline{M}$ is the equatorial hypersphere of M. The equation (3) reduces to the ordinary differential equation

$$\rho''(u)=k\rho+b$$

along the ρ-curves, see [4].

The assumption of the theorem means $V\neq\phi$.

Case (I) of $U=\phi$. Then $M=N_1\cup N_2$, and we suppose $N_2\neq\phi$. In a connected component V_0 of $V\cap N_2$, we have the equation

$$(4) \qquad \nabla_j\rho_i=c^2\rho g_{ji}$$

c being a positive constant, and

$$(5) \qquad \Phi=\rho_i\rho^i=c^2\rho^2.$$

In an adapted coordinate system in V_0, ρ is given by $\rho=ae^{cu}$, a being a constant. It follows from this expression and the differentiability of ρ that $M=N_2$ and the equation (4) is valid on the whole manifold M.

If $U\neq\phi$, then it is proved that Φ is the sum

$$(6) \qquad \Phi=\rho_\lambda\rho^\lambda=\Phi_1+\Phi_2$$

of functions Φ_1 of (x^i) and Φ_2 of (x^p), and the parts Φ_1 and Φ_2 satisfy the equations

$$(7) \qquad \nabla_j\nabla_i(\Phi_1-k\rho^2)=\Omega g_{ji}, \quad \nabla_q\nabla_p(\Phi_2+k\rho^2)=\Omega g_{qp},$$

where we have put

$$\Omega=k(\Phi_1-\Phi_2-k\rho^2)+b,$$

b being a constant. Moreover we have the equations

$$(8) \qquad \begin{cases} \nabla_p\nabla_j\nabla_i\rho^2=\nabla_p(\Phi_2+k\rho^2)g_{ji}, \\ \nabla_i\nabla_q\nabla_p\rho^2=\nabla_i(\Phi_1-k\rho^2)g_{qp}, \end{cases}$$

$$(9) \qquad \begin{cases} \nabla_k\nabla_j\nabla_i\rho^2=\nabla_k(\Phi_1+k\rho^2)g_{ji}+g_{kj}\nabla_i\Phi_1+g_{ki}\nabla_j\Phi_1, \\ \nabla_r\nabla_q\nabla_p\rho^2=\nabla_r(\Phi_2-k\rho^2)g_{qp}+g_{rq}\nabla_p\Phi_2+g_{rp}\nabla_q\Phi_2, \end{cases}$$

and

$$(10) \qquad \begin{cases} \nabla_k\nabla_j\nabla_i\Phi_1=k(2g_{ji}\nabla_k\Phi_1+g_{kj}\nabla_i\Phi_1+g_{ki}\nabla_j\Phi_1), \\ \nabla_r\nabla_q\nabla_p\Phi_2=-k(2g_{qp}\nabla_r\Phi_2+g_{rq}\nabla_p\Phi_2+g_{rp}\nabla_q\Phi_2). \end{cases}$$

Case (II) where $k=0$ in U. The equations (8) and (9) together make the tensor equation

$$\nabla_\nu\nabla_\mu\nabla_\lambda\rho^2=g_{\nu\mu}\nabla_\lambda\Phi+g_{\nu\lambda}\nabla_\mu\Phi+g_{\mu\lambda}\nabla_\nu\Phi \tag{11}$$

and the equations (7) do

$$\nabla_\mu\nabla_\lambda\Phi=bg_{\mu\lambda}. \tag{12}$$

This case splits into three cases.

(a) $b=0$ and Φ is constant in U. Noting (6), we can obtain the equation

$$\nabla_\mu\nabla_\lambda\rho^2=2\Phi g_{\mu\lambda}.$$

Referring this equation to an adapted coordinate system (u, u^α) for ρ^2, and choosing suitably the arc-length u, we obtain the expression

$$\rho^2=\Phi u^2. \tag{13}$$

(b) $b=0$ and Φ is not constant. Then we have $\nabla_\mu\nabla_\lambda\Phi=0$. Integrating the equation (11) in an adapted one for Φ, we can see that this case does not occur locally.

(c) $b\neq 0$ in some connected component of U. In an adapted coordinate system (u, u^α) for Φ, ρ^2 is expressed as

$$\rho^2=\frac{1}{8b}[(bu^2+4\gamma)^2+4f^{\alpha\beta}\gamma_\beta\gamma_\alpha], \tag{14}$$

where γ is a solution of the equation

$$\bar\nabla_\gamma\bar\nabla_\beta\bar\nabla_\alpha\gamma=-(2f_{\beta\alpha}\bar\nabla_\gamma\gamma+f_{\gamma\beta}\bar\nabla_\alpha\gamma+f_{\gamma\alpha}\bar\nabla_\beta\gamma) \tag{15}$$

in an $(n-1)$-dimensional manifold $\bar M$ with metric tensor $f_{\beta\alpha}$.

Case (III) where $k\neq 0$ in some component U_0 of U. By means of (7), $\Phi_1-k\rho^2$ and $\Phi_2+k\rho^2$ are special concircular scalar fields in $M_1\cap U_0$ and $M_2\cap U_0$. We may put $k=c^2$, $c>0$. Referring (7)–(9) to adapted coordinate systems (u, u^α) in $M_1\cap U_0$ and (v, v^ξ) in $M_2\cap U_0$, $\alpha=2, \cdots, n_1$, $\xi=n_1+2, \cdots, n$, and noting (6), we obtain the expressions of ρ^2 in the forms

$$\rho^2=\begin{cases} \text{(a)} & \dfrac{1}{c^2}\left(\omega_1 e^{2cu}-\omega_2\sin^2 cv+\dfrac{A}{c^2}e^{cu}\cos cv+B\right), \\ \text{(b)} & \dfrac{1}{c^2}\left(\omega_1\cosh^2 cu-\omega_2\sin^2 cv+\dfrac{A}{c^2}\sinh cu\cos cv+B\right), \\ \text{(c)} & \dfrac{1}{c^2}\left(\omega_1\sinh^2 cu-\omega_2\sin^2 cv+\dfrac{A}{c^2}\cosh cu\cos cv+B\right), \end{cases} \tag{16}$$

according to the forms of solution of (7), where A, B are constants, ω_1 is a function of u^α and ω_2 a function of v^ξ satisfying certain equations similar to (15).

By means of the expressions (16), we see that, in any case of the above, the sets N_1 and N_2 are border sets in M, the constants appearing in (16) are common with all components of U and the expressions are valid over the whole manifold M.

If M is complete, then so are the parts M_1 and M_2, in particular,

M_2 is 1-dimensional or an n_2-sphere, and γ in Case (I, c) and ω_2 in Case (III) are constant or bounded.

For example, we treat Case (III, a). Let P be a point of U, $M_1(P)$ the part passing through P, Γ a ρ-curve of the restriction $(\Phi_1 - c^2\rho^2)|M_1(P)$, $\Gamma^* = f(\Gamma)$, and s^* the arc-length of Γ^*. Then ω_1 on Γ should be positive and we put $\omega_1 = 2a^2$. We take a value u_0 so large that $\rho > (a/c)e^{cu}$ for $u > u_0$, and s_0^* the value corresponding to u_0. Then we obtain the inequality

$$s^* - s_0^* < \frac{1}{a} e^{-cu_0}.$$

Hence the length of Γ^* is bounded as $u \to \infty$. This contradicts to the globalness of the conformal diffeomorphism $f: M \to M^*$.

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