60. A Uniqueness Theorem in an Identification Problem for Coefficients of Parabolic Equations

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1. **Introduction and results.** In this paper we consider the parabolic equation

$$\frac{\partial u}{\partial t}+\left(p(x)-\frac{\partial^2}{\partial x^2}\right)u=0 \qquad (\text{in } (0,\infty)\times\Omega)$$

under the Neumann boundary condition

$$\frac{\partial}{\partial n}u=0 \qquad (\text{on } (0,\infty)\times\partial\Omega),$$

with the initial condition

$$u|_{t=0}=a(x) \qquad (x\in\Omega),$$

where $\Omega=(0,1)\subset R^1$. In what follows, however, the coefficient $p(x)$ and the initial value $a(x)$ are to be determined, while values of the solution on the boundary $u(t,\xi)$, $(\xi\in\partial\Omega)$, are regarded as observed and known functions of $t\in[T_1,T_2]$ for some T_1,T_2 with $0<T_1<T_2<\infty$. Namely, we are concerned with the following

Problem. Can we determine $\{p,a\}$ through $\{u(t,\xi);T_1<t<T_2,\ \xi=0,1\}$?

It is obvious that the answer is negative without any assumption on $\{p,a\}$. Actually, if $a=0$, then $u\equiv 0$ for any p. Hence we introduce the following

Definition. The realization in $L^2(\Omega)$ of the differential operator $p(x)-\partial^2/\partial x^2$ with the Neumann boundary condition is denoted by A_p. The eigenvalues and eigenfunctions of A_p are denoted by $\{\lambda_n;n=1,2,\cdots\}$ and $\{\phi(\cdot,\lambda_n);n=1,2,\cdots\}$, respectively. Then, an initial value $a\in L^2(\Omega)$ is said to be a generating element with respect to A_p iff $(a,\phi(\cdot,\lambda_n))\neq 0$ for any n, where $(\ ,\ )$ is the L^2-inner product.

Then we have the following

Theorem. *Consider the following equations* (I) *and* (II) *for* $p,q\in C^1[0,1]$. *Let* $a,b\in L^2(0,1)$ *and assume that* a *is a generating element with respect to* A_p:

$$\text{(I)}\qquad \begin{cases}\dfrac{\partial u}{\partial t}+\left(p(x)-\dfrac{\partial^2}{\partial x^2}\right)u=0 & (0<t<\infty,\ x\in(0,1))\\[2mm] \left.\dfrac{\partial u}{\partial x}\right|_{x=0,1}=0 & (0<t<\infty)\\[2mm] u|_{t=0}=a(x) & (x\in(0,1)),\end{cases}$$

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\text{(II)} \quad \begin{cases} \dfrac{\partial v}{\partial t}+\left(q(x)-\dfrac{\partial^2}{\partial x^2}\right)v=0 & (0<t<\infty,\ x\in(0,1)) \\ \dfrac{\partial v}{\partial x}\Big|_{x=0,1}=0 & (0<t<\infty) \\ v|_{t=0}=b(x) & (x\in(0,1)). \end{cases}
$$

Then, the equality $u(t,\xi)=v(t,\xi)$ $(T_1<t<T_2,\ \xi=0,1)$ *implies* $p(x)=q(x)$ $(x\in[0,1])$ *and* $a(x)=b(x)$ (a.e. $x\in(0,1)$).

Remark 1. Pierce [7] considered a similar parabolic equation with null initial condition, with non-homogeneous boundary condition of the third kind on $\xi=0$ and with homogeneous boundary condition of the same kind on $\xi=1$. He showed that in such a case, the values $u(t,\xi)$ $(0<t<T_1,\ \xi=0)$ determines the spectral density function of A_p, whence $p(x)=q(x)$ follows by the theory of Gel'fand-Levitan [2]. In proving our Theorem, we are also inspired by Gel'fand-Levitan's idea of using a certain integral operator to transform eigenfunctions. However, our method is rather direct and does not use their theory itself.

Remark 2. Recently, one of the authors has succeeded in constructing $\{p, a\}$ theoretically in terms of $\{u(t,\xi)\,;\ T_1<t<T_2,\ \xi=0,1\}$. His method is more heavily based on Gel'fand-Levitan's theory. According to his result it is necessary for $a(x)$ to be a generating element with respect to A_p, in order that $\{p, a\}$ should be uniquely determined by $\{u(t,\xi)\,;\ T_1<t<T_2,\ \xi=0,1\}$. Detailed discussions of these results are given in Murayama [5] along with some extensions to other problems such as determination of the coefficients of $A_\alpha=-(\partial/\partial x)(\alpha(x))(\partial/\partial x)$, or of the boundary conditions.

Remark 3. As for other works concerning inverse problems for parabolic equations we refer to Sabatier [9], Prilenko [8], Isakov [3], Iskenderov [4] and Chavent [1].

2. Outline of the proof of Theorem. The realization in $L^2(\Omega)$ of the operator $q(x)-\partial^2/\partial x^2$ with the Neumann boundary condition is denoted by A_q, and its eigenvalues and eigenfunctions are denoted by $\{\mu_m\}$ and $\{\psi(\cdot,\mu_m)\}$, respectively. We normalize $\{\phi(\cdot,\lambda_n)\}$ and $\{\psi(\cdot,\mu_m)\}$ as $\phi(0,\lambda_n)=\psi(0,\mu_m)=1$ $(n, m=1,2,\cdots)$. Then solutions u, v are given by the following eigenfunction-expansion:

$$
(1) \qquad u(t,x)=\sum_{n=1}^{\infty} e^{-\lambda_n t}(a,\phi(\cdot,\lambda_n))/\rho_n\cdot\phi(x,\lambda_n)
$$

$$
(2) \qquad v(t,x)=\sum_{m=1}^{\infty} e^{-\mu_m t}(b,\psi(\cdot,\mu_m))/\sigma\cdot\psi(x,\mu_m),
$$

where $\rho_n=\int_0^1\phi(x,\lambda_n)^2dx$ and $\sigma_m=\int_0^1\psi(x,\mu_m)^2dx$. By the hypothesis we have $u(t,\xi)=v(t,\xi)$ $(T_1<t<T_2,\ \xi=0,1)$, which holds for any t in $0<t<\infty$ by analytic continuation in t:

$$(3)\quad \sum_{n=1}^{\infty} e^{-\lambda_n t}(a, \phi(\cdot, \lambda_n))/\rho_n \cdot \phi(\xi, \lambda_n) = \sum_{m=1}^{\infty} e^{-\mu_m t}(b, \psi(\cdot, \mu_m))/\sigma_m \cdot \psi(\xi, \mu_m) \qquad (0<t<\infty, \xi=0, 1).$$

Since $(a, \phi(\cdot, \lambda_n)) \neq 0$ and λ_n is simple $(n=1, 2, \cdots)$, we have, by putting $\xi=0$,

$$(4)\qquad \lambda_n = \mu_{m(n)}$$

and

$$(5)\qquad (a, \phi(\cdot, \lambda_n))/\rho_n = (b, \psi(\cdot, \mu_{m(n)}))/\sigma_{m(n)} \neq 0$$

for some $m(n)$. On the other hand, it is well known that

$$\lambda_n^{1/2} = n\pi + 0\left(\frac{1}{n}\right) \qquad (n \to \infty)$$

$$\mu_m^{1/2} = m\pi + 0\left(\frac{1}{m}\right) \qquad (m \to \infty).$$

This means that $m(n)=n$ in (4) and (5). Moreover, by putting $\xi=1$ in (3), we have

$$(6)\qquad \phi(1, \lambda_n) = \psi(1, \mu_n) \qquad (n=1, 2, \cdots).$$

We need the following

Lemma 1. *There exists a C^2-class function $K=K(x, y)$ in $0 \leqq y \leqq x \leqq 1$ subject to*

$$(\mathrm{E})\qquad \begin{cases} K_{xx}(x, y) - K_{yy}(x, y) + p(y)K(x, y) = q(x)K(x, y) \\ K_y(x, 0) = 0 \\ K(x, x) = \dfrac{1}{2}\displaystyle\int_0^x \{q(s) - p(s)\} ds. \end{cases}$$

Lemma 2. *With $K(x, y)$ in the preceding lemma, the eigenfunctions are related to each other as*

$$(7)\qquad \psi(x, \mu_n) = \phi(x, \lambda_n) + \int_0^x K(x, y)\phi(y, \lambda_n) dy \qquad (n=1, 2, \cdots).$$

Proof of Lemma 2. We denote the right hand side of (7) by $\psi(x)$. By use of $(p(x) - (d^2/dx^2))\phi(x, \lambda_n) = 0$, $\phi'(0, \lambda_n) = 0$, $\phi(0, \lambda_n) = 1$ and (E), we obtain

$$(8)\qquad \lambda\psi(x) + \frac{d^2}{dx^2}\psi(x) = q(x)\psi(x) \qquad (\lambda = \lambda_n),$$

$$(9)\qquad \psi(0) = 1$$

and

$$(10)\qquad \psi'(0) = 0.$$

Indeed, (9) is immediate, and (10) is obvious from

$$\psi'(x) = \phi'(x, \lambda_n) + K(x, x)\phi(x, \lambda_n) + \int_0^x K_x(x, y)\phi(y, \lambda_n) dy.$$

Finally, (8) is verified as

$$\left(\lambda_n + \frac{d^2}{dx^2}\right)\psi(x) = \left(\frac{d^2}{dx^2} + \lambda_n\right)\phi(x, \lambda_n) + \left(\frac{d}{dx}K(x, x) + K_x(x, x)\right)\phi(x, \lambda_n) + K(x, x)\phi'(x, \lambda_n) + \int_0^x K(x, y)\lambda_n\phi(y, \lambda_n) dy$$

$$=p(x)\phi(x,\lambda_n)+\Big(\frac{d}{dx}K(x,x)+K_x(x,x)\Big)\phi(x,\lambda_n)$$
$$+K(x,x)\phi'(x,\lambda_n)+\int_0^x K(x,y)\Big[p(y)-\frac{d^2}{dy^2}\Big]\phi(y,\lambda_n)dy$$
$$=p(x)\phi(x,\lambda_n)+\Big(\frac{d}{dx}K(x,x)+K_x(x,x)+K_y(x,x)\Big)\phi(x,\lambda_n)$$
$$-K_y(x,0)+\int_0^x [K_{xx}(x,y)+K(x,y)p(y)-K_{yy}(x,y)]\phi(y,\lambda_n)dy$$
$$=q(x)\psi(x) \qquad (\because(\mathrm{E})).$$

Uniqueness of the solution of the Cauchy problem of (8), (9) and (10) implies (7). Q.E.D.

Now, by (6), $\psi'(1,\lambda_n)=0$ and (7), we have

(11) $$\int_0^1 K(1,y)\phi(y,\lambda_n)dy=0 \qquad (n=1,2,\cdots)$$

and

(12) $$K(1,1)\phi(1,\lambda_n)+\int_0^1 K_x(1,y)\phi(y,\lambda_n)dy=0 \qquad (n=1,2,\cdots).$$

Therefore, the completeness of $\{\phi(\cdot,\lambda_n); n=1,2,\cdots\}$ implies $K(1,y)=K_x(1,y)=0$ $(y\in[0,1])$. By considering the domain of dependence of the hyperbolic equation in (E), we have $K(x,y)=0$ $(0\leqq 1-x\leqq y\leqq x\leqq 1)$, so that we have $p(x)=q(x)$ $(1/2\leqq x\leqq 1)$ by the last equality of (E). Now, by transforming x to $\tilde{x}=1-x$ and repeating the same argument as above, we have $p(x)=q(x)$ $(0\leqq x\leqq 1/2)$, whence follows $p\equiv q$. Therefore, (5) implies $(a,\phi(\cdot,\lambda_n))=(b,\phi(\cdot,\lambda_n))$ $(n=1,2,\cdots)$, hence we obtain $a(x)=b(x)$ (a.e. $x\in(0,1)$). Q.E.D.

Proof of Lemma 1. This lemma can be proved in some standard way as we sketch below. We extend the coefficients p and q to $p\in C^1[-1,1]$ and $q\in C^1[0,2]$ and construct the solution $K=K(x,y)$ of (E) in $\{(x,y); |x-1|+|y|<1\}$. By transforming the variables (x,y) to (X,Y) as $X=(1/2)(x+y)$ and $Y=(1/2)(x-y)$, we seek the solution $k=k(X,Y)$ of the following system of equations (E′) on $[0,1]\times[0,1]$:

(E′) $$\begin{cases}\dfrac{\partial^2}{\partial X\partial Y}k(X,Y)=r(X,Y)k(X,Y)\\ \Big(\dfrac{\partial k}{\partial X}-\dfrac{\partial k}{\partial Y}\Big)(X,X)=0\\ k(X,0)=f(X),\end{cases}$$

where $k(X,Y)=K(X+Y,X-Y)$, $r(X,Y)=1/2\{q(X+Y)-p(X-Y)\}$ and $f(X)=1/2\int_0^x \{q(s)-p(s)\}ds$. Let $R(X,Y;X_0,Y_0)$ be the Riemann's function of the hyperbolic equation $(\partial^2/\partial X\partial Y)k=r(X,Y)k$ (see, e.g., Picard [6]). Putting $Q(X,Y)=(\partial R/\partial X-\partial R/\partial Y)(Y,Y;X,0)$, we can show that $Q(X,Y)$ is in $C^2([0,1]\times[0,1])$ and that the equation

(G) $$g(X)+\int_0^X Q(X,Y)g(Y)dY=2f(X)-f(0)$$

has a solution $g \in C^2([0,1]\times[0,1])$. Furthermore, we can show that the system of equations

$$(\mathrm{E}'') \qquad \begin{cases} \dfrac{\partial^2}{\partial X \partial Y} k = r(X, Y)k \\ k(X, X) = g(X) \\ k(X, 0) = f(X) \end{cases}$$

has a solution $k = k(X, Y)$ in $C^2([0,1]\times[0,1])$ and that $k = k(X, Y)$ is a solution of (E′) simultaneously. In fact, the second equation of (E′) is obtained by differentiating

$$\int_0^X R(Y, Y; X, 0)\left(\frac{\partial k}{\partial Y} - \frac{\partial k}{\partial X}\right)(Y, Y)dY = 0,$$

which follows from the Riemann's formula

$$\begin{aligned} k(X, 0) = &\frac{1}{2}\{k(X, X) + k(0, 0)\} \\ &+ \frac{1}{2}\int_0^X \left\{R(Y, Y; X, 0)\left(\frac{\partial k}{\partial Y} - \frac{\partial k}{\partial X}\right)(Y, Y)\right. \\ &\left. + \left(\frac{\partial R}{\partial X} - \frac{\partial R}{\partial Y}\right)(Y, Y; X, 0)k(Y, Y)\right\}dY. \end{aligned}$$

Q.E.D.

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