108. *The Lax-Milgram Theorem for Banach Spaces. I*

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§ 0. When V is a Hilbert space over $\boldsymbol{R}$ and 'a' is a symmetric, continuous, coercive bilinear form, the Lax-Milgram theorem is an immediate consequence of the Riesz representation theorem for Hilbert spaces. However, the case when 'a' is no longer symmetric is different. In this paper, we present a method in § 1, which treats the non-symmetric case also almost on the same lines as the symmetric case. The method gives actually the Lax-Milgram theorem for any Banach space. The idea behind the method also generalizes to Banach spaces, the theorem of Lions-Stampacchia [1] on variational inequalities, proved by them for Hilbert spaces.

§ 1. Let V be a vector space over $\boldsymbol{R}$. Let 'a' be a bilinear form on V such that $a(x,x)>0\ \forall x\neq 0$. Let '$b$' be the bilinear form defined as

$$b(x,y)=\frac{a(x,y)+a(y,x)}{2}\forall x,y\in V.$$

Then, 'b' is symmetric and $b(x,x)=a(x,x)\forall x\in V$. Hence, '$b$' defines an inner-product on V and endowed with this inner-product, V becomes a pre-Hilbert space which we denote by V_b. We shall denote by $\|x\|$, the norm of an element $x\in V_b$. i.e. $\|x\|=+\sqrt{a(x,x)}$. Let V_b' denote the dual of V_b.

Let us assume that 'a' is continuous on $V_b\times V_b$. i.e. let us assume that $\exists M<+\infty$ such that

$$|a(x,y)|\leqslant M\sqrt{a(x,x)}\sqrt{a(y,y)} \qquad \forall x,y\in V.$$

Then, under this assumption, we have obvious linear maps A and B from V_b to V_b' taking an element $x\in V_b$ to $Ax\in V_b'$ (resp. $Bx\in V_b'$) defined as $Ax(y)=a(y,x)$ (resp. $Bx(y)=a(x,y)$).

$$\|Ax\|=\sup_{y\neq 0}\frac{|Ax(y)|}{\|y\|}=\sup_{y\neq 0}\frac{|a(y,x)|}{\|y\|}\leqslant M\|x\|.$$

Moreover, if $x\neq 0$,

$$\|Ax\|\geqslant\frac{a(x,x)}{\|x\|}=\|x\|.$$

Hence, if $x\neq 0$,

$$\|x\|\leqslant\|Ax\|\leqslant M\|x\|.$$

But these inequalities are trivially valid when $x=0$. Hence, we have $\forall x\in V$,

$$\|x\| \leqslant \|Ax\| \leqslant M\,\|x\|. \tag{I}$$

We have, similarly $\|x\| \leqslant \|Bx\| \leqslant M\,\|x\|\ \forall x \in V$.

Definition 1. Let 'a' be continuous on $V_b \times V_b$. V_b is said to have the *right* (resp. *left*) Riesz representation property with respect to 'a' if $\forall f \in V_b'$, $\exists x \in V$ such that $f(y) = a(y, x)$ (resp. $f(y) = a(x, y)$) $\forall y \in V$.

In terms of the maps A, B, V_b has the right (resp. left) Riesz representation property iff A (resp. B) is onto. From the inequalities (I), A and B are one-one. Hence, there is always uniqueness of the element x, that corresponds to $f \in V_b'$ in the above definition.

Theorem 1. *Let 'a' be continuous on $V_b \times V_b$. Then, V_b has the right (resp. left) Riesz representation property with respect to 'a' iff V_b is complete i.e. iff V_b is a Hilbert space.*

Proof. We shall prove the theorem for the right Riesz representation property. The proof for the left Riesz representation property is similar.

(i) *Necessity.* Let us assume that V_b has the right Riesz representation property with respect to 'a'. This means A is an isomorphism of V_b and V_b'. Because of the inequalities (I), A is a topological isomorphism too. But V_b' is always complete as the dual of any normed space over $\boldsymbol{R}$ or $\boldsymbol{C}$ is always complete. Hence, V_b is also complete.

(ii) *Sufficiency.* Let us assume that V_b is complete. We have to prove that $A(V_b) = V_b'$. Suppose not, then $\exists f \in V_b'$ such that $f \notin A(V_b)$. Since V_b is complete, the inequalities (I) show that $A(V_b)$ is a closed subspace of V_b'. Hence, by the Hahn-Banach theorem, $\exists \beta \in V_b''$, the double dual of V_b such that β vanishes on $A(V_b)$, but $\beta(f) \neq 0$. Since V_b is complete and hence is a Hilbert space, it is reflexive. Therefore, β is given by an element of V_b. i.e. $\exists u \in V$ such that $\beta(h) = h(u)\,\forall h \in V_b'$. Thus, $\exists$ an element $u \in V$ such that $f(u) \neq 0$, but $a(u, v) = 0\ \forall v \in V$. But $a(u, v) = 0\ \forall v \in V \Rightarrow a(u, u) = 0$ in particular, which in turn implies that $u = 0$. But this contradicts the fact $f(u) \neq 0$. Hence, $A(V_b) = V_b'$, proving that V_b has the right Riesz representation property with respect to 'a'.

Corollary (Lax-Milgram theorem). *Let $(V, \|\ \|)$ be a Banach space over $\boldsymbol{R}$. Let 'a' be a continuous bilinear form on V which is coercive. i.e. $\exists \delta > 0$ such that $a(x, x) \geqslant \delta \|x\|^2\ \forall x \in V$. Then, $\forall f \in V'$, the dual of $(V, \|\ \|)$, $\exists$ a unique $u \in V$ (resp. unique $w \in V$) such that $f(v) = a(v, u)$ (resp. $f(v) = a(w, v)$) $\forall v \in V$.*

Proof. Since 'a' is coercive, $a(x, x) > 0\ \forall x \neq 0$. The continuity and coercivity of 'a' imply that $(V, \|\ \|)$ and V_b are isomorphic. Hence, 'a' is continuous on $V_b \times V_b$ and V_b is complete. Therefore, by Theorem 1, V_b has both right and left Riesz representation properties with respect to 'a'. From this, the corollary follows immediately by observing that $f \in V' \Leftrightarrow f \in V_b'$. Q.E.D.

The idea behind the proof of the Lax-Milgram theorem is, we first prove it for the space V_b on which 'a' is trivially coercive, by assuming V_b is complete and 'a' continuous on $V_b \times V_b$. This is Theorem 1. Then, we are able to prove the theorem immediately for the Banach space $(V, \|\ \|)$ on which 'a' is continuous and coercive, as $(V, \|\ \|)$ then becomes isomorphic to V_b.

The same idea helps us to generalize the result of Lions-Stampacchia [1] on variational inequalities to Banach spaces. They proved the theorem for Hilbert spaces.

Theorem 2 (Lions-Stampacchia). *Let $(V, \|\ \|)$ be a Banach space over $\boldsymbol{R}$. Let 'a' be a continuous, bilinear form on V. Then, given any closed convex set K and any $f \in V'$, $\exists$ a unique $u \in K$ such that*

$$a(u, v-u) \geqslant f(v-u) \forall v \in K.$$

Proof. Since 'a' is continuous and coercive on $(V, \|\ \|)$, $(V, \|\ \|)$ and V_b are isomorphic. Therefore, 'a' is continuous on $V_b \times V_b$ and V_b is a Hilbert space. Further, 'a' is trivially coercive on V_b. Hence, the theorem of Lions-Stampacchia applies in this case. Thus, for any closed convex set L of V_b and any $f \in V_b'$, $\exists$ a unique $u \in L$ such that $f(v-u) \leqslant a(u, v-u)\ \forall v \in L$. From this, Theorem 2 follows immediately by observing that $(V, \|\ \|)$ and V_b have the same dual and the same closed convex sets. Q.E.D.

Reference

[1] Lions-Stampacchia: Variational inequalities. Comm. Pure Appl. Math., **20**, 493–519 (1967).