82. *A Generalization of Poincaré Normal Functions on a Polarized Manifold*

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1. Recently J. L. Dupont found out the connection between continuous cohomologies of semi-simple Lie groups and integrals of invariant forms over geodesic simplices in symmetric spaces ([5]). In this note we shall study the analytic structure of analogous integrals of rational forms over a simplex-like polyhedron which more or less corresponds to an n-th iterated path, associated with $(n+1)$ intersection points of n-ple hyperplane sections in a polarized manifold. It will be shown that *these can be expressed by means of a finite sum of iterated integrals of special* 1*-forms in the sense of K. T. Chen*, which can be regarded as a natural generalization of abelian integrals on projective algebraic varieties ([8]). *The notion of periods of abelian integrals will also be generalized as the part of corresponding "shuffle structures" fixed by monodromy groups.*

2. Let (V, E) be an n-dimensional polarized complex manifold. Let $|E|$ be the complete linear system of Cartier divisors associated with the line bundle E. We denote by h the dimension of $H^0(V, \mathcal{O}_V(E))$. Consider the space $X=X_m$ consisting of sequences of m linearly independent sections $s_1, s_2, \cdots, s_m$ of $H^0(V, \mathcal{O}_V(E))$. X_m is isomorphic to the $\mathrm{Stief}_{m,h}$, the space of sequences of m linearly independent vectors in $\boldsymbol{C}^h$. Let $S_1, S_2, \cdots, S_m$ be m Cartier divisors in $|E|$, associated with $s_1, s_2, \cdots, s_m$, respectively. We shall call this a "*configuration of hyperplane sections*" and the set of all them "*configuration space of hyperplane sections*". This is parametrized by X_m.

Let W be an algebraic subset of V of codimension 1 such that $V-W$ is affine if W is not empty. We denote by $\Omega^{\cdot}(V, *W)$ the space of rational forms on V with poles in W. Let $S_{-n}, S_{-n+1}, \cdots, S_0$ be $(n+1)$ Cartier divisors in $|E|$ such that $S_{-n}, S_{-n+1}, \cdots, S_0$ and W are in general position.

Definition 1. Let v_i, $-n \leqq i \leqq 0$, be arbitrary points of $S_{-n} \cap S_{-n+1} \cap \cdots \cap S_{i-1} \cap S_{i+1} \cap \cdots \cap S_{-1} \cap S_0$. We consider a simplex-like n-polyhedron Δ of class C^1 disjoint from W, satisfying the following conditions: i) $\partial \Delta_{i_1, i_2, \cdots, i_p} = \bigcup_{j \notin \{i_1, \cdots, i_p\}} \Delta_{j, i_1, i_2, \cdots, i_p}$ where $\Delta_{i_1, i_2, \cdots, i_p}$ denote $\Delta \cap S_{i_1} \cap \cdots \cap S_{i_p}$. ii) $\Delta_{-n, \cdots, i-1, i+1, \cdots, 0}$ consists of the only one point v_i.

This will be called a "*fundamental simplex with the vertices* $v_0, v_1, \cdots, v_{-n}$".

By making use of the isotopy theorem due to R. Thom, it can be easily seen that such a Δ can be constructed from lower dimensional faces.

We consider the relative analytic space $\mathfrak{X}$ consisting of pairs $(V - W, S_{-n} \cup S_{-n+1} \cup \cdots \cup S_0)$, $\langle S_{-n}, \cdots, S_0 \rangle \in X$, so that we have the natural projection $\pi: \mathfrak{X} \mapsto X$, with the fibre $(V-W, S_{-n} \cup \cdots \cup S_0)$. Let Y be the subset of X such that π becomes singular, namely the configuration $\langle S_{-n}, \cdots, S_0 \rangle$ and W are not in general position. Then $\mathfrak{X} - \pi^{-1}(Y)$ is a topological fibre bundle over $X-Y$ with the above fibre.

Now we are interested in the analytic structure of the integral

(1) $$\tilde{\eta} = \int_{\Delta} \eta, \qquad \text{for } \eta \in \Omega^n(V, *W).$$

Lemma 1. *η being fixed, $\tilde{\eta}$ depends only on the homotopy class of Δ, provided that v_i, $-n \leqq i \leqq 0$, are all fixed. Namely let $\Delta(t)$, $0 \leqq t \leqq 1$, be a continuous family of Δ such that $\Delta_{i_1, i_2, \cdots, i_p}(t) \subset V_{i_1, i_2, \cdots, i_p} = S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}$ and $\Delta_{i_1, i_2, \cdots, i_n}(t)$ are fixed. Then $\tilde{\eta}$ is independent of t.*

For the proof see, for example, [9].

We put $\hat{\Omega}_I = \oplus_{0 \leqq q \leqq n-p} \hat{\Omega}_I^q$ for the ordered sequence $I = (i_1, i_2, \cdots, i_p)$ where $\hat{\Omega}_I^q$ denotes $\oplus_{J \supset I} \Omega^{q-|J-I|}(V_J, *(W \cap V_J))$. When I is empty, we denote $\hat{\Omega}_\phi$ simply by $\hat{\Omega}$. Let ε_I be the canonical projection from $\hat{\Omega}$ onto $\hat{\Omega}_I$. We can define boundary operators $\hat{d}$ and $\hat{d}_I$ on $\hat{\Omega}$ and $\hat{\Omega}_I$, respectively, as follows:

(2) $$(\hat{d}\varphi)_{i_1, i_2, \cdots, i_p} = d(\varphi_{i_1, i_2, \cdots, i_p}) + \sum_{q=1}^{p} (-1)^{q-1} \cdot \varphi_{i_1, \cdots, i_{q-1}, i_{q+1}, \cdots, i_p}, \qquad \text{for } \varphi = (\varphi_{i_1, i_2, \cdots, i_p})_{0 \leqq p \leqq n} \in \hat{\Omega},$$

on each V_I. Then the following is commutative:

(3) $$\begin{array}{ccc} \hat{\Omega} & \xrightarrow{\hat{d}} & \hat{\Omega} \\ \downarrow{\scriptstyle \varepsilon_I} & & \downarrow{\scriptstyle \varepsilon_I} \\ \hat{\Omega}_I & \xrightarrow{\hat{d}_I} & \hat{\Omega}_I \end{array}.$$

Then we have an extended de Rham complex $(\hat{\Omega}, \hat{d})$ *with the nilpotent covariant derivation $\hat{d}$, associated with the configuration* $\langle S_{-n}, S_{-n+1}, \cdots, S_0 \rangle$. We denote by $C(V_I)$ the cell complex in V_I over $\boldsymbol{C}$. Let $\hat{C} = \oplus_{0 \leqq p \leqq n} \hat{C}_p$, $\hat{C}_p = \oplus_I C_{p-|I|}(V_I)$ be the chain complex with the boundary operation:

(4) $$(\hat{\partial} c)_{i_1, i_2, \cdots, i_p} = c_{i_1, i_2, \cdots, i_p} - \sum_{j \notin \{i_1, i_2, \cdots, i_p\}} c_{j, i_1, i_2, \cdots, i_p} (-1)^p$$

in V_I for $c = (c_I) \in \hat{C}_n$.

We now define the natural pairing between $\hat{\Omega}$ and $\hat{C}$ as follows:

(5) $$\langle \varphi, c \rangle = \sum_I \int_{c_I} \varphi_I.$$

Then we have the Stokes formula:

(6) $$\langle \hat{d}\varphi, c \rangle = \langle \varphi, \hat{\partial} c \rangle.$$

The integral $\tilde{\eta}$ can be regarded as an element of $H^n(\hat{\Omega}, \hat{d})$, by taking as $\varphi_\phi = \eta$ and $\varphi_I = 0$ otherwise. Δ itself becomes a cycle.

Proposition 1. *$H^n(\hat{\Omega}, \hat{d})$ has a filtration F_I satisfying the following conditions:* i) $F_I = H^*(\hat{\Omega}_I, \hat{d}_I)$, ii) $F_I \supset F_J$ *if* $I \subset J$, *and* iii) $H^{n-|I|}(\hat{\Omega}_I / \sum_{J \supset I} \hat{\Omega}_J, \hat{d}_I) = F_I \cap H^{n-|I|}(\hat{\Omega}_I, \hat{d}_I) / \sum_{J \supset I} F_J \cap H^{n-|I|}(\hat{\Omega}_I, \hat{d}_I)$ $= H^{n-|I|}(V_I - W \cap V_I, C)$.

We denote by $H^0(X, \Theta(*Y))$ the space of rational vector fields on X with poles only on Y. Then

Proposition 2. *For any $\tau \in H^0(X, \Theta(*Y))$, the covariant differentiation ∇ of the Gauss-Manin connection:*

$$\left\langle \tau, d_X \int_c \varphi \right\rangle = \int_c \nabla_\tau \varphi \tag{7}$$

acting on $\mathcal{O}_{X-Y} \cdot H^*(\hat{\Omega}, \hat{d})$, satisfies

$$\nabla_\tau \mathcal{O}.F_I \subset \mathcal{O}.F_I \oplus \textstyle\sum_{J \supsetneqq I} \mathcal{O}.F_J. \tag{8}$$

This follows from the following

Lemma 2. *Let V be an affine variety of dimension n embedded in C^{n+m}. Let $f_0, f_1, \cdots, f_n$ be linearly independent linear functions on C^{n+m}. Let Δ be an n-polyhedron in V satisfying $\partial\Delta = \bigcup_{i=0}^n \partial\Delta \cap \{f_i = 0\}$. We assume that each f_j depends holomorphically on t in an open neighbourhood $U \subset C$. Then*

$$d/dt \int_\Delta \eta = \int_\Delta \frac{\partial \eta}{\partial t} + \sum_{j=0}^n \int_{\partial\Delta \cap \{f_j=0\}} \partial f_j / \partial t \cdot \eta / df_j \tag{9}$$

for a holomorphic n-form η on Δ.

According to Proposition 1, there exists a basis $\{e_I^{(\nu)}, 1 \leqq \nu \leqq \mu_I\}$ of $H^{n-|I|}(V_I - V_I \cap W, C)$ such that each $\{e_J^{(\nu)}; 1 \leqq \nu \leqq \mu_J, J \supset I\}$ forms a basis of $H^{n-|I|}(\hat{\Omega}_I, \hat{d}_I)$. Let P_I be a system of μ_I linearly independent horizontal solutions of the Gauss-Manin connection D_I on $H^{n-|I|}(\hat{\Omega}_I / \sum_{J \supset I} \hat{\Omega}_J, \hat{d}_I)$ $= H^{n-|I|}(V_I - V_I \cap W, C)$. Then there exists an integrable connection form $\omega_I = (\omega_{I,s}^r) \in \Omega^1(X, *Y) \otimes gl(\mu_I, C)$ such that

$$D_I P_I = d_X P_I - \omega_I \cdot P_I = 0. \tag{10}$$

According to Proposition 2 we have

$$d_X \int e_I^{(r)} - \sum_{s=1}^{\mu_I} \omega_{I,s}^r \int e_I^s = \sum_{J \supsetneqq I, s=1}^{\mu_I} A_{(I,J),s}^r \int e_J^{(s)} \tag{11}$$

with $A_{(I,J),s}^r(x, dx) \in \Omega^1(X, *Y)$. Therefore by solving the differential equation (11), we arrive at the following

Theorem 1. *For any sequence $\phi \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \{-n, -n+1, \cdots, 0\}$, the integral $\tilde{\eta}$, being a linear combination of $\int e_\phi^{(r)}$, $1 \leqq r \leqq \mu_\phi$, can be described as an element of the $\Omega^0(X, *Y)$-module generated by the $\mu_\phi \cdot \mu_{I_n}$ components of the matrix valued iterated integrals of the following type:*

$$P_\phi(x) \cdot \int^x P_\phi^{-1}(x_1) \cdot A_{\phi I_1}(x_1, dx_1) \cdot P_{I_1}(x_1) \cdot \int^{x_1} P_{I_1}^{-1}(x_2) \cdot A_{I_1, I_2}(x_2, dx_2) \tag{12}$$

$$\times P_{I_2}(x_2)\cdot\int^{x_2}\cdots\int^{x_{n-1}} P^{-1}_{I_{n-1}}(x_n)\cdot A_{I_{n-1},I_n}(x_n, dx_n)\cdot P_{I_n}(x_n).$$

According to K. T. Chen's formula (see [4, p. 222]) we have

Corollary. *The monodromy M_γ, $\gamma\in\pi_1(X-Y,*)$ preserves each F_I: $M_\gamma\cdot F_I\subset F_I$. Using the dual basis $\{e^*_{J,r}\}$ of the above $\{e^r_J\}$, M_γ can be written in an explicit way:*

(13) $$M_\gamma(e^*_{I,r})=\sum_{J\supset I,s=1}^{\mu_J} M^s_{(J,I),r}\cdot e^*_{J,s}.$$

Therefore M_γ is unipotent if and only if $M_{(J,J)}$ are all the identities.

By taking a suitable finite covering $\tilde{X}$ of X, we may assume that $M_{(I_n,I_n)}$ and $M_{(\phi,\phi)}$ are the identities of orders $\deg(V,E)$ and $\dim H^n(V-W,C)$, respectively. The fixed part $\mathrm{Hom}_C(H^n(\hat{\Omega},\hat{d}),C)^{\pi_1}$ of $\pi_1(X-Y,*)$-module $\mathrm{Hom}_C(H^n(\hat{\Omega},\hat{d}),C)$ contains $H_n(V-W,C)$ when $V-W$ is affine and contains the $(n,0)$-part of $H_n(V,C)$ when W is empty. When n is equal to 1, this coincides with the usual periods system of abelian integrals. Under this situation the following questions seem interesting: *Do $H_n(V-W,C)$ and the $(n,0)$-part of $H_n(V,C)$ coincide with $\mathrm{Hom}_C(H^n(\hat{\Omega},\hat{d}),C)^{\pi_1}$ when $V-W$ are affine and empty respectively? Does the totality of elements of the matrices $M_{(I_n,\phi)}\in\mathrm{Hom}(Z[\pi_1(\tilde{X}-\tilde{Y},*),R^{\mu_\phi\mu_{I_n}})$ generate $\mathrm{Hom}_C(H^n(\hat{\Omega},\hat{d}),C)^{\pi_1}$? It also seems interesting to give any relation between $\mathrm{Hom}_C(H^n(\hat{\Omega},\hat{d}),C)^{\pi_1}$ and Griffiths intermediate Jacobian* (see [7]).

3. In this section we shall give important examples where M_γ are all unipotent. From now on we shall assume *the Fujita Δ-genus $\Delta(V,E)$ vanishes.* Then it is known that (V,E) is isomorphic to a) the complex projective space (CP^n,H), b) the hyper-quadric (Q^n,H), c) the tautological line bundle of an ample vector bundle over the projective line and its base space, or d) (CP^2,H^2) where H denotes the hyperplane bundle (see [6]). We shall take as W the union of Cartier divisors $S_1,S_2,\cdots,S_m$ of $|E|$ in general position. Then we have

Proposition 3. *There exists a finite covering $(\tilde{X},\tilde{Y})$ over (X,Y) branched along Y such that*

(14) $$\nabla_\tau\mathcal{O}.F_I\subset\sum_{J\supsetneqq I}\mathcal{O}.F_J$$

for any $\tau\in H^0(X,\Theta(*Y))$. $M_{(J,J)}$ all become the identities.

Actually ∇_τ can be explicitely computed (see also [1]).

Definition 2. Consider the space $B^0(\Omega^\cdot(\tilde{X},\log\langle\tilde{Y}\rangle))$ spanned by iterated integrals on the path space $\mathcal{P}(\tilde{X}-\tilde{Y},*)$ of $\tilde{X}-\tilde{Y}$:

(15) $$\int\omega_{i_1},\omega_{i_2},\cdots,\omega_{i_p}$$

where $\omega_j\in\Omega^1(\tilde{X},\log\langle\tilde{Y}\rangle)$. The elements of B^0 depending only on homotopy classes in $\mathcal{P}(\tilde{X}-\tilde{Y},*)$ will be called "*hyper-logarithms of p-th order*" (see [2]).

Then Proposition 2 implies immediately the following

Theorem 2. *If $\Delta(V,E)=0$, then the integral $\tilde{\eta}$ can be described as a finite sum of*

(rational functions)$\times$(hyper-logarithms of at most n-th order) on $\tilde{X}$ with singularities only on $\tilde{Y}$.

In view of Lemma 2, Proposition 2 can be proved case by case, by computing suitable bases of the cohomologies $H^{n-|I|}(V_I - V_I \cap W, \boldsymbol{C})$. (It is essential that all $\Delta(V_I, E_I)$ vanish for $E_I = E|_{V_I}$.) In fact, by using a technique in [3], we have

Lemma 3. Case a) *We put $V' = V - S_m$ and $W' = V' \cap W$. Then W' is the union of hyperplane sections $S_j : f_j = 0$ $(1 \leqq j \leqq m-1)$ in general position in $V' = \boldsymbol{C}^n$. As is well known, $H^n(V' - W', \boldsymbol{C})$ has a basis consisting of the logarithmic forms:*

$$d\log f_{i_1} \wedge \cdots \wedge d\log f_{i_n}.$$

Case b) *Let V' and W' as above. Then W' is the union of hyperplane sections $S_j : f_j = 0$ $(1 \leqq j \leqq m-1)$ in the hyperquadric $V' : x_0^2 + x_1^2 + \cdots + x_n^2 = 1$ in $\boldsymbol{C}^{n+1}$. $H^n(V' - W', \boldsymbol{C})$ has a basis:*

$$\frac{\theta}{f_{i_1} f_{i_2} \cdots f_{i_p}},\ 0 \leqq p \leqq n, \quad \text{and} \quad \frac{\{f_0, f_{i_1}, \cdots, f_{i_n}\}^{\perp}}{f_{i_1} f_{i_2} \cdots f_{i_n}} \theta,\ p = n,$$

with $1 \leqq i_1 < \cdots < i_p \leqq m-1$ and $\theta = \sum_{j=0}^{n} (-1)^j \cdot x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n$, where $\{f_0, f_{i_1}, \cdots, f_{i_n}\}^{\perp}$ denotes a non-zero linear function g such that $(g, 1) = (g, f_{i_1}) = \cdots = (g, f_{i_n}) = 0$, and (a, b) denotes $\sum_{j=0}^{n+1} \alpha_j \beta_j$ for $a = \sum_{j=0}^{n} \alpha_j x_j + \alpha_{n+1}$ and $b = \sum_{j=0}^{n} \beta_j x_j + \beta_{n+1}$.

Case c) *There exists a sequence of positive integers $\mu_1, \mu_2, \cdots, \mu_n$ such that V is embedded in $\boldsymbol{C}P^{h-1}$, $h = \mu_1 + \mu_2 + \cdots + \mu_n + n$, by the mapping*

$$\begin{array}{ccc} \boldsymbol{C}^2 \times \boldsymbol{C}^n & \to & \boldsymbol{C}P^{h-1} \\ \cup\!\!\!| & & \cup\!\!\!| \\ (w_0, w_1; \zeta_1, \zeta_2, \cdots, \zeta_n) & \to & (u_{j,k}) \end{array}$$

where $u_{j,k} = w_0^{\mu_j - k} \cdot w_1^k \cdot \zeta_j$. Let S_{m+1} be the divisor defined by $w_0 = \zeta_1 = 0$ in V which is in general position with respect to $S_1, S_2, \cdots, S_m$. Then $V' = V - S_{m+1}$ is isomorphic to $\boldsymbol{C}^n$ with the coordinates $w_1/w_0 = x_1, \zeta_2/\zeta_1 = x_2, \cdots, \zeta_n/\zeta_1 = x_n$. Let W' be the union of hypersurfaces $S_j : f_j = 0$ in V', $1 \leqq j \leqq m$, where $f_j = \sum_{k=2}^{n} \alpha_{jk}(x_1) \cdot x_k + \alpha_{j1}(x_1)$, $\alpha_{jk}(x_1) \in \boldsymbol{C}[x_1]$. $H^n(V' - W', \boldsymbol{C})$ has a basis

$$\frac{x_1^\sigma}{[i_1, i_2, \cdots, i_{n-1}]} dx_1 \wedge d\log f_{i_1} \wedge d\log f_{i_2} \wedge \cdots \wedge d\log f_{i_{n-1}},$$

$1 \leqq i_1 < \cdots < i_{n-1} \leqq m$, $0 \leqq \sigma \leqq \deg\,[i_1, i_2, \cdots, i_{n-1}] - 1$ and

$$x_1^\sigma \cdot \frac{dx_1 \wedge \cdots \wedge dx_n}{f_{i_1} \cdots f_{i_n}}$$

$1 \leqq i_1 < \cdots < i_n \leqq m$, $0 \leqq \sigma \leqq \deg\,[i_1, i_2, \cdots, i_n] - 1$, where $[i_1, i_2, \cdots, i_{n-1}]$ and $[i_1, i_2, \cdots, i_n]$ denote the determinants

$$\begin{vmatrix} \alpha_{i_1,2}\cdots\alpha_{i_1,n} \\ \cdots \\ \cdots \\ \alpha_{i_{n-1},2}\cdots\alpha_{i_{n-1},n} \end{vmatrix} \quad \textit{and} \quad \begin{vmatrix} \alpha_{i_1,1}\cdots\alpha_{i_1,n} \\ \cdots \\ \cdots \\ \alpha_{i_n,1}\cdots\alpha_{i_n,n} \end{vmatrix}$$

respectively.

Case d) *Let S_{m+1} be the line at infinity in CP^2, which is in general position with respect to $S_1, S_2, \cdots, S_m$. Let V' be $CP^2 - S_{m+1} = C^2$ Let W' be the union of $S_j : f_j = 0$. Then $H^n(V' - W', C)$ has a basis*

$$\varphi_{ij}(x_1, x_2)\frac{df_i \wedge df_j}{f_i f_j} \quad \textit{and} \quad \frac{dx_1 \wedge dx_2}{f_i}$$

where $\varphi_{ij}(x_1, x_2) \in C[x_1, x_2]$ mod. *the ideal (f_i, f_j).*

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