# 83. A Formula for the Dimension of Spaces of Siegel Cusp Forms of Degree Three

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**0.** In this note we obtain a dimension formula for the vector spaces of Siegel cusp forms of degree three. Our main theorem is the following

**Theorem.** *The dimension of the vector space of Siegel cusp forms of weight $k\geqq 5$ with respect to $\Gamma_3(l)$ $(l\geqq 3)$ is equal to*

$$(2^{-16}3^{-6}5^{-2}7^{-1}l^{21}(2k-2)(2k-3)(2k-4)^2(2k-5)(2k-6)-2^{-10}3^{-2}5^{-1}l^{16}(2k-4)+2^{-8}3^{-3}l^{15})\Pi_{p\mid l\ p:prime}(1-p^{-2})(1-p^{-4})(1-p^{-6}).$$

Details and proofs will be published elsewhere.

**1.** Let $\mathfrak{S}_g$ be the Siegel upper half plane of degree $g$, $\Gamma_g(l)$ the principal congruence subgroup of the Siegel modular group of level $l(l\geqq 3)$, $\mathfrak{S}_g^*(l)$ the quotient complex analytic space of $\mathfrak{S}_g$ by $\Gamma_g(l)$, and let $\bar{\mathfrak{S}}_g^*(l)$ and $\tilde{\mathfrak{S}}_g^*(l)$ be the Satake and the Voronoi compactification of $\mathfrak{S}_g^*(l)$, respectively, which are constructed in [7] and in [6]. $\tilde{\mathfrak{S}}_g^*(l)$ is non-singular if $g\leqq 4$ and there exists a morphism $s:\tilde{\mathfrak{S}}_g^*(l)\to\bar{\mathfrak{S}}_g^*(l)$ which is identity on $\mathfrak{S}_g^*(l)$.

Let $L_g$ be the line bundle on $\mathfrak{S}_g^*(l)$ determined by the automorphic factor on $\mathfrak{S}_g$:

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\mapsto \det(CZ+D), \quad M\in \mathrm{Sp}(g, \boldsymbol{Z}) \quad \text{and} \quad Z\in\mathfrak{S}_g.$$

$L_g$ is extended to $\bar{\mathfrak{S}}_g^*(l)$ as a line bundle by Siegel's $\Phi$-operator, which we denote by $\bar{L}_g$. Let $\tilde{L}_g$ be the pullback of $\bar{L}_g$ to $\tilde{\mathfrak{S}}_g^*(l)$. Put $\Delta(g)=\tilde{\mathfrak{S}}_g^*(l)-\mathfrak{S}_g^*(l)$, which is a divisor with simple normal crossings, if $g\leqq 4$. Then we have a vanishing theorem:

**Theorem.** $H^p(\tilde{\mathfrak{S}}_g^*(l), \mathcal{O}(k\tilde{L}_g-\Delta(g)))=0$, *if* $g\leqq 4$, $l\geqq 3$, $k\geqq g+2$, *and* $p>0$.

Since the vector space of Siegel cusp forms of degree $g$ and weight $k$ with respect to $\Gamma_g(l)$ is isomorphic to $H^0(\tilde{\mathfrak{S}}_g^*(l), \mathcal{O}(k\tilde{L}_g-\Delta(g)))$, by this theorem and Riemann-Roch-Hirzebruch's theorem the dimension of this space is written as a sum of intersection numbers.

**2.** Let $\bar{X}$ be an $n$-dimensional complex manifold and $\Delta$ a reduced divisor with simple normal crossings on $\bar{X}$. Put $X=\bar{X}-\Delta$.

**Definition.** Let $\Theta_{\bar{X}}(\log\Delta)$ be the subsheaf of $\Theta_{\bar{X}}$ which is generated by $z_1\frac{\partial}{\partial z_1}, \cdots, z_l\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_{l+1}}, \cdots, \frac{\partial}{\partial z_n}$ at $x \in \bar{X}$, where we chose a system of local coordinates $(z_1, \cdots, z_n)$ around $x$ so that $\varDelta$ is defined by $z_1 \cdots z_l = 0$ around $x$. $\Theta_X(\log \varDelta)$ is dual to $\Omega^1_X(\log \varDelta)$ over $\mathcal{O}_X$ which is defined in [2]. Put $\bar{c}_j(X) = c_j(\Theta_X(\log \varDelta))$ which we call *the $j$-th logarithmic chern class of $X$ in $\bar{X}$.*

Let $\varDelta$ be as above and put $\varDelta = \bigcup D_i$ where each $D_i$ is an irreducible divisor on $X$. We always denote a divisor, its cohomology class, and the line bundle associated with the divisor by the same symbol. Let $\varDelta_k$ be the $k$-th fundamental symmetric function of $D_i$'s for $k \geqq 1$ and $\varDelta_0 = 1$, and let $c_j(\bar{X})$ be the $j$-th chern class of $\bar{X}$. Then

**Proposition.** $c_j(\bar{X}) = \Sigma_{k=0}^{j} \bar{c}_{j-k}(X) \cdot \varDelta_k$.

By the relation $(g+1)\tilde{L}_g = -\bar{c}_1(g)$ and this proposition, the dimension of the vector space of Siegel cusp forms is written as a polynomial of $\bar{c}_j(g)$'s and $\varDelta_k$'s if $g \geqq 4$, where $\bar{c}_j(g)$ is the $j$-th logarithmic chern class of $\mathfrak{S}^*_g(l)$ in $\tilde{\mathfrak{S}}^*_g(l)$. The terms which do not include $\varDelta_k$ are calculated by the following theorem. We identify any cohomology class in $H^{g(g+1)}(\tilde{\mathfrak{S}}^*_g(l), \boldsymbol{Z})$ with its value at the fundamental cycle.

**Theorem** ([5]). *Let $v$ be an invariant measure on $\mathfrak{S}_g$, then*

$$\bar{c}_{j_1}(g) \cdots \bar{c}_{j_k}(g) = a(j_1, \cdots, j_k) v(\mathfrak{S}_g / \Gamma_g(l)),$$

*where $j_1 + \cdots + j_k = 2^{-1}g(g+1)$ and $a(j_1, \cdots, j_k)$ is a constant depending only on $v$.*

**3.** The following lemma is an analogue of the adjunction formula.

**Lemma.** *Let $X$, $\bar{X}$, and $\varDelta$ be as above, and let $\bar{D} \subset \bar{X} - X$ be an irreducible divisor and $\bar{D}_1, \bar{D}_2, \cdots$ other irreducible divisors contained in $\bar{X} - X$. Put $D = \bar{D} - \bigcup_{i \geqq 1} \bar{D}_i$, then*

$$\bar{c}_j(X) \,|\, \bar{D} = \bar{c}_j(D) \ (j = 0, \cdots, n),$$

where the right hand side is the logarithmic chern class of $D$ in $\bar{D}$.

Let $X \subset \tilde{\mathfrak{S}}^*_3(l) - \mathfrak{S}^*_3(l)$ be an irreducible divisor, then $X$ is a fiber space over $\bar{\mathfrak{S}}^*_2(l)$ and this fibering factors through $s : \tilde{\mathfrak{S}}^*_2(l) \to \bar{\mathfrak{S}}^*_2(l)$ as a equi-dimensional morphism $\pi : X \to \tilde{\mathfrak{S}}^*_2(l)$. The general fiber of $\pi$ is a 2-dimensional abelian variety. The following theorem is an analogue of the canonical line bundle formula of elliptic surfaces.

**Theorem.** *There exists a cohomology class $e_j$ on $\tilde{\mathfrak{S}}^*_2(l)$ such that*

$$\bar{c}_j(3)|_X = \pi^*(e_j) \ (j = 0, \cdots, 6).$$

This theorem is proved by the above lemma and the following exact sequence of locally free sheaves:

$$0 \to \pi^* \Omega^1_{\tilde{\mathfrak{S}}^*_2(l)}(\log \varDelta(2)) \to \Omega^1_X(\log \varDelta(3)' \cap X) \to \Omega^1_{X/\tilde{\mathfrak{S}}^*_2(l)}(\log \varDelta(3)' \cap X) \to 0,$$

where $\varDelta(3)'$ is the closure of $\varDelta(3) - X$ in $\tilde{\mathfrak{S}}^*_3(l)$.

By this theorem the vanishing of intersection numbers such as $\bar{c}_1(3)^5 X$ and $\bar{c}_5(3) X$ is proved.

**4.** Let $T$ be an $n$-dimensional algebraic torus over complex number field and $T\subset X$ a non-singular torus embedding (cf. [4]). Torus embeddings are classified by "rational partial polyhedral decomposition" in $\boldsymbol{R}^n$. Let $D_1, D_2, \cdots$ be irreducible divisors on $X$ contained in $X-T$ and $\gamma_i$ the 1-dimensional cone in $\boldsymbol{R}^n$ corresponding to $D_i$, and let $\boldsymbol{a}_i=(a_{i1},\cdots,a_{in})$ be the primitive vector which spanns $\gamma_i$, where $\boldsymbol{a}_i$ is said to be primitive if and only if $a_{ij}(j=1,\cdots,n)$ are integers and relatively prime. Then

**Theorem.** $\Sigma_i a_{ij}\cdot D_i\sim 0$ $(j=1,\cdots,n)$, *where* $\sim$ *means the linear equivalence. Note that the left hand side is locally a finite sum even if* $X$ *is not of finite type.*

$\mathfrak{S}_3^*(l)$ is constructed by gluing copies of quotients of torus embeddings and if $X_1, X_2, X_3\subset\tilde{\mathfrak{S}}_3^*(l)-\mathfrak{S}_3^*(l)$ are irreducible divisors, then $X_1\cap X_2\cap X_3$ is contained in a single quotient of a torus embedding if and only if $s(X_1\cap X_2\cap X_3)$ is 0-dimensional. In this case we can calculate intersection numbers such as $X_1^2X_2^2X_3^2$, etc. by this theorem.

**5.** Let $X$ be as in § 3 and $Y\subset\tilde{\mathfrak{S}}_2^*(l)-\mathfrak{S}_2^*(l)$ an irreducible divisor, then it follows that

$$\tilde{L}_{3|X}\pi^*(Y)^4[X]=\pi^*(\tilde{L}_2Y^4)[X]=0,$$

since $\tilde{\mathfrak{S}}_2^*(l)$ is 3-dimensional. Although the intersection number

$$\pi^*(\tilde{L}_2Y^2)(X_{|X})^2[X]$$

does not vanish, we can calculate this by the result in § 4. From these equations as above we can derive relations among $\tilde{L}_3X_1^3X_2^2$ and $\tilde{L}_3X_1^4X_2$, etc., where $X_1, X_2\subset\tilde{\mathfrak{S}}_3^*(l)-\mathfrak{S}_3^*(l)$ are irreducible divisors and we can calculate them. Similarly we can derive relations among $X_1^3X_2^3$, $X_1^4X_2^2$, and $X_1^5X_2$, etc., but to calculate them we lack one relation.

**6.** To calculate the remaining intersection numbers, we use the results of [3] concerning theta constants.

**Definition.** Let $\tau\in\mathfrak{S}_g$, then $\tau$ is said to be reducible if and only if $\tau$ is equivalent with respect to $\Gamma_g(1)$ to the following point:

$$\begin{bmatrix}\tau_2 & 0\\ 0 & \tau_1\end{bmatrix},\quad \tau_2\in\mathfrak{S}_{g-1}\quad\text{and}\quad \tau_1\in\mathfrak{S}_1.$$

There exist thirty six theta constants of degree three.

**Theorem.** $\tau\in\mathfrak{S}_3$ *is reducible if and only if at least two theta constants vanish at* $\tau$.

**Theorem.** *Let* $\chi_{18}$ *be the product of thirty six theta constants and* $\Sigma_{140}$ *the* 35*-th fundamental symmetric function of the* 8*-th powers of thirty six theta constants. Then* $\chi_{18}$ *and* $\Sigma_{140}$ *are Siegel cusp forms with respect to* $\Gamma_3(1)$ *of weight* 18 *and* 140, *respectively.*

Let $I$ and $J$ be the zero loci of the sections of $18\tilde{L}_3$ and $140\tilde{L}_3$ determined by $\chi_{18}$ and $\Sigma_{140}$, respectively, and let $\bar{I}$ and $\bar{J}$ be the respective closures of $I$ and $J$ in $\tilde{\mathfrak{S}}_3^*(l)$. Then we have

$$18\tilde{L}_3\sim\bar{I}+2l\Delta(3),$$

and

$$140\tilde{L}_3 \sim \bar{J} + 15l\Delta(3).$$

Let $R_g$ be {reducible points of $\mathfrak{S}_g\}/\Gamma_g(l)$ and $\bar{R}_g$ its closure in $\tilde{\mathfrak{S}}_g^*(l)$, and let $X$ be as in § 3. We put $R(X)=\pi^{-1}(\bar{R}_2)$ and $R(\Delta(3))=$ the union of $R(X)$, when $X$ moves irreducible divisors in $\tilde{\mathfrak{S}}_3^*(l)-\mathfrak{S}_3^*(l)$. Then $\bar{I}\cdot\bar{J}$ has support at $\bar{R}_3$ by the above theorem and it is seen that $\bar{I}\cdot\bar{J}$ also has support at $R(\Delta(3))$ by considering the limits of theta constants when $\tau$ tends to $\Delta(3)$.

Therefore we obtained the following equation:

$$(18\tilde{L}_3-2l\Delta(3))(140\tilde{L}_3-15l\Delta(3))=\bar{I}\cdot\bar{J}=n_1\bar{R}_3+n_2R(\Delta(3)),$$

where $n_1$ and $n_2$ are the multiplicity of $\bar{I}\cdot\bar{J}$ at $\bar{R}_3$ and $R(\Delta(3))$. By calculating $n_1$ and $n_2$, and replacing $R(\Delta(3))$ by the rational equivalence we have the following

**Theorem.** $(3\Delta_1(3)^2+\Delta_2(3))l^2=24\bar{R}_3+60l\tilde{L}_3\Delta_1(3)-252\tilde{L}_3^2$, *where* $\Delta_2(3)$ *is defined as in* § 2.

Let $X_1, X_2$ be as in § 5, then it is easily seen that $\bar{R}_3X_1^2X_2^2=0$. Therefore multiplying the equation in the theorem by $X_1^2X_2^2$, we derive the relation among intersection numbers which we lacked in § 5 and we can calculate them. Let $X$ be as above. We can calculate $\tilde{L}_3^3X^3$ by the result in § 4, and we can calculate $\bar{R}_3\tilde{L}_3^2X^2$, $\bar{R}_3\tilde{L}_3X^3$, and $\bar{R}_3X^4$. Hence multiplying the equation in the theorem by $\tilde{L}_3^2X^2$, $\tilde{L}_3X^3$, and $X^4$, we can calculate $\tilde{L}_3^2X^4$, $\tilde{L}_3X^5$, and $X^6$. Especially $\tilde{L}_3^2X^4$ vanishes.

By [1, § 16.4] and [5], there exists a relation: $15\bar{c}_1^2(3)=32\bar{c}_2(3)$. Therefore we have $\bar{c}_2(3)X^4=0$.

Thus we calculated the all intersection numbers which appear in the Riemann-Roch-Hirzebruch's formula and from this result the main theorem is proved.

7. Now we shall discuss the problem to obtain the dimension formula of Siegel cusp forms of degree $g\geqq 4$. It is not known that the Voronoi compactification of degree $g\geqq 5$ is non-singular or not. Therefore we can hope that we can obtain the dimension formula only of degree four. But there exist three difficulties.

First we do not have the result to write the set of reducible points of $\mathfrak{S}_4$ as zeros of Siegel cusp forms. To obtain this result it is necessary to solve the following

**Conjecture.** $\tau\in S_4$ is reducible if and only if at least three theta constants vanish at $\tau$.

In [3] the theorem to write $R_3$ as zeros of theta constants is obtained by the theory of curves. Since the point of $\mathfrak{S}_4$ does not correspond to a Jacobian variety of genus 4 in general, the theory of curves will be no longer applied. But in the case of degree three, we did not used the family of Jacobian varieties of genus 3 over $\tilde{\mathfrak{S}}_3^*(l)$, but the family of Jacobian varieties of genus 2 over $\tilde{\mathfrak{S}}_2^*(l)$ which appears in the boundary of $\tilde{\mathfrak{S}}_3^*(l)$. In the case of degree four we use the family of Jacobian varieties $\pi: X \to \tilde{\mathfrak{S}}_3^*(l)$ of genus three. Therefore it suffices to know the common zeros in $X$ of the limits of three theta constants of degree four when $\tau$ tends to $X$, which is weeker problem than the above conjecture and we can hope that the theory of curves can be applied to this problem.

Secondly the fibering of $X$ above is not equi-dimensional.

Thirdly there existed a relation between $\bar{c}_1(3)$ and $\bar{c}_2(3)$. In the case degree four, $\bar{c}_j(4)$ $(j \geqq 3)$ appears in the Riemann-Roch-Hirzebruch's formula and $\bar{c}_j(4)$ $(j \geqq 3)$ is not proportional to $\bar{c}_1(4)^j$.

Even if these difficulties are overcome, it will be necessary to accomplish a rather tiresome process of calculation.

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