74. A Note on Modular Forms mod $\mathfrak{p}$

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Let N be a positive integer and χ be a Dirichlet character mod N. Let $\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c\equiv 0 \pmod{N}\right\}$. Let $f(z)$ be a cusp form of weight k satisfying

$$f(\sigma(z))=(cz+d)^k\chi(d)f(z) \qquad \text{for all } \sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Then we call $f(z)$ a cusp form of type (k,χ) on $\Gamma_0(N)$, and we denote by $S_k(N,\chi)$ the space of all cusp forms of type (k,χ) on $\Gamma_0(N)$.

From now we fix a prime number $p, p\geq 5$. Let N be a positive integer such that $(p,N)=1$. Let ψ and χ be any Dirichlet characters mod N and mod p respectively such that $\psi\chi(-1)=1$. Let t be the order of χ and put $\kappa=\dfrac{(p-1)(t-a)}{t}$ with an integer a such that $1\leq a\leq t$ and $(a,t)=1$. Let k be any even positive integer. Then we can prove the following simple identities between dimensions of spaces of cusp forms by using Hijikata's trace formula [1]:

Theorem 1. *The notations being as above, we have*

$$\dim_{\boldsymbol{C}} S_k(Np,\psi\chi)=\dim_{\boldsymbol{C}} S_{(p+1)k/2-\kappa}(N,\psi)+\dim_{\boldsymbol{C}} S_{(p+1)k/2-(p-1-\kappa)}(N,\psi).$$

As an application of Theorem 1, we can study some properties of cusp forms mod p in the sense of Serre and Swinnerton-Dyer.

We fix our notations. We may fix N, ψ and k. Take an algebraic number field K of finite degree over the rational number field which contains all eigenvalues of all Hecke operators acting on $S_k(Np,\psi\chi)$ for all Dirichlet characters χ mod p and on $S_{k'}(N,\psi)$ for all $k'\leq\dfrac{k}{2}(p+1)$, and p-th roots of unity. We fix a prime divisor $\mathfrak{p}$ of K lying over p. Let ν be the normalized valuation of K attached to $\mathfrak{p}$ so that $\nu(p)=p^{-1}$ and $\mathfrak{o}=\{\alpha\in K\mid \nu(\alpha)\leq 1\}$, $F=\mathfrak{o}/\mathfrak{p}$.

For any Dirichlet character χ mod p, let

$$V_\chi=\left\{f(z)=\sum_{n=1}^{\infty} a_n q^n \in S_k(Np,\psi\chi) \mid a_n\in K \quad \text{for all } n\geq 1\right\},$$

$$V_{k'}=\left\{g(z)=\sum_{n=1}^{\infty} b_n q^n \in S_{k'}(N,\psi) \mid b_n\in K \quad \text{for all } n\geq 1\right\},$$

where $q=e^{2\pi iz}$. V_χ and $V_{k'}$ are vector spaces over K with same dimensions as those of $S_k(Np,\psi\chi)$ and $S_{k'}(N,\psi)$ over the complex number field.

Let V be any subspace of vector spaces defined as above. We define

$$V(\mathfrak{o})=\left\{f(z)=\sum_{n=1}^{\infty} a_n q^n \in V \mid a_n \in \mathfrak{o} \quad \text{for all } n \geq 1\right\}.$$

For any $f=\sum_{n=1}^{\infty} a_n q^n$ in $V(\mathfrak{o})$, we define a formal power series $\tilde{f}$ in $F[[q]]$ by

$$\tilde{f}=\sum_{n=1}^{\infty} \bar{a}_n q^n \quad \text{where} \quad \bar{a}_n = a_n \bmod \mathfrak{p}.$$

$\tilde{f}$ is a cusp form mod p in the slightly generalized sense than that of Serre and Swinnerton-Dyer. Put $\tilde{V}=\{\tilde{f} \mid f \in V(\mathfrak{o})\}$, then $\tilde{V}$ is the vector space over F with the same dimension as that of V.

Let $W_p=\begin{pmatrix} px, & 1 \\ pNy, & p \end{pmatrix}$, with some integers x and y, be a matrix with determinant p. We define a linear operator on V_χ by

$$(f \mid W_p)(z)=p^{k/2}(pNyz+p)^{-k} f\left(\frac{pxz+1}{pNyz+p}\right).$$

Then W_p gives an isomorphism between V_χ and $V_{\bar{\chi}}$ where $\bar{\chi}$ is the Dirichlet character mod p such that $\bar{\chi}(n)$=the complex conjugate of $\chi(n)$. Since we fix a prime divisor $\mathfrak{p}$, there exists a unique Dirichlet character ω mod p such that $\omega(a) \equiv a \bmod \mathfrak{p}$ for all $a \in \boldsymbol{Z}$, $(a, p)=1$. Hence if χ is a Dirichlet character mod p with order t, there exists a unique integer a, $1 \leq a \leq t$, $(a, t)=1$ such that

$$\chi=\omega^{-(p-1)(t-a)/t}. \quad \text{Put } \kappa=\frac{(p-1)(t-a)}{t}.$$

Theorem 2. *The notations being as above, there is a decomposition of V_χ into a direct sum of subspaces $V_{1,\chi}$ and $V_{2,\chi}$ satisfying following properties:*

1) $\dim_K V_{1,\chi}=\dim_C S_{(p+1)k/2-(p-1-\kappa)}(N, \psi)$,
   $\dim_K V_{2,\chi}=\dim_C S_{(p+1)k/2-\kappa}(N, \psi)$.
2) $\widetilde{V_{1,\chi}}=\widetilde{V_{(p+1)k/2-(p-1-\kappa)}}$.
3) $(\widetilde{V_{2,\chi} \mid W_p})=\widetilde{V_{(p+1)k/2-\kappa}}$.

This is a generalization of Theorem 11 in [2].

As a corollary of Theorem 2, we have

Corollary. *For any Hecke operator $T(n)$ of degree n, we have the following congruence:*

$$\begin{aligned}\operatorname{tr} T(n) \quad \text{on} \quad S_k(Np, \psi\chi) \equiv \operatorname{tr} T(n) \quad \text{on} \quad & S_{(p+1)k/2-(p-1-\kappa)}(N, \psi) \\ +n^{\kappa} \cdot \operatorname{tr} T(n) \quad \text{on} \quad & S_{(p+1)k/2-\kappa}(N, \psi) \qquad (\bmod \mathfrak{p}).\end{aligned}$$

Detailed proofs of Theorems 1 and 2 and other applications of these results will appear elsewhere.

Note. Dr. K. Hatada informed us that he had already got similar results to Theorems 1 and 2 in the case $K=2$, ψ=the trivial character mod N and χ=the trivial character mod p in his doctoral thesis at University of Tokyo, April 1979.

References

[1] H. Hijikata: Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$. J. Math. Soc. Japan, **26**, 56–82 (1974).

[2] J.-P. Serre: Formes modulaires et fonctions zêta p-adiques, Modular functions of one variable. III. Proc. Intern. Summer School, Univ. Antwerp. Lect. Notes in Math., vol. 350, Springer, pp. 191–268 (1972).