69. *Some Properties of Non-Commutative Multiplication Rings*

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In this short note we shall discuss some properties of non-commutative multiplication rings, especially non-idempotent multiplication rings. Commutative multiplication rings were studied by S. Mori in [3], [4], and also in his earlier works. We denote $A \subseteq B$ if A is a subset of B, and by $A < B$ if A is a proper subset of B. We do not assume the existence of the identity, and "ideal" means a two-sided ideal.

1. Multiplication rings. Definition. A ring R is called a *multiplication ring* or briefly *M-ring*, if for any ideal $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{a} < \mathfrak{b}$, there exist ideals $\mathfrak{c}, \mathfrak{c}'$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c} = \mathfrak{c}'\mathfrak{b}$.

Proposition 1. *Let R be an M-ring, let $\mathfrak{p}$ be a proper prime ideal, and let $\mathfrak{q}$ be any ideal properly containing $\mathfrak{p}$, then $\mathfrak{p}\mathfrak{q} = \mathfrak{q}\mathfrak{p} = \mathfrak{p}$.*

Proof. Since $\mathfrak{p} < \mathfrak{q}$, there exist ideals $\mathfrak{b}, \mathfrak{b}'$ such that $\mathfrak{p} = \mathfrak{q}\mathfrak{b} = \mathfrak{b}'\mathfrak{q}$, therefore $\mathfrak{p} \subseteq \mathfrak{b}$. On the other hand $\mathfrak{q}\mathfrak{b} \equiv 0 \pmod{\mathfrak{p}}$, $\mathfrak{q} \not\equiv 0 \pmod{\mathfrak{p}}$, implies $\mathfrak{b} \equiv 0 \pmod{\mathfrak{p}}$, hence $\mathfrak{p} = \mathfrak{b}$, and similarly $\mathfrak{p} = \mathfrak{b}'$.

Proposition 2. *Let R be an M-ring, and let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals such that $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$, then $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$.*

Proof. Since $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$, $\mathfrak{p}_2 < (\mathfrak{p}_1, \mathfrak{p}_2)$, therefore by Proposition 1 $\mathfrak{p}_2 = \mathfrak{p}_2(\mathfrak{p}_1, \mathfrak{p}_2) = (\mathfrak{p}_2\mathfrak{p}_1, \mathfrak{p}_2^2)$. If $\mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1$, then we have $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$, which contradicts our assumptions, therefore $\mathfrak{p}_2\mathfrak{p}_1 < \mathfrak{p}_1$, hence there exists an ideal $\mathfrak{c}$ such that $\mathfrak{p}_2 \supseteq \mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{c}$, and $\mathfrak{p}_1 \not\equiv 0 \pmod{\mathfrak{p}_2}$, therefore $\mathfrak{c} \equiv 0 \pmod{\mathfrak{p}_2}$. Thus we have $\mathfrak{p}_2\mathfrak{p}_1 \subseteq \mathfrak{p}_1\mathfrak{p}_2$. In a similar way we have $\mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_2\mathfrak{p}_1$, therefore $\mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{p}_2$.

Theorem 1. *Let R be an M-ring, then the multiplication of prime ideals is commutative.*

Proof. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals of R. If $\mathfrak{p}_1 < \mathfrak{p}_2$, then by Proposition 1 $\mathfrak{p}_1 = \mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{p}_2$. $\mathfrak{p}_2 < \mathfrak{p}_1$ implies the same results. If $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$, then by Proposition 2 $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$.

2. Non-idempotent M-ring. Definition. *An M-ring R such that $R > R^2$ is called a non-idempotent M-ring.*

Theorem 2. *Let R be non-idempotent M-ring, and let $\mathfrak{a}$ be an ideal of R, then $\mathfrak{a} = R^\rho$ for some positive integer ρ or $\mathfrak{a} \subseteq \bigcap_{n=1}^{\infty} R^n$.*

Proof. Let $\mathfrak{a}$ be an ideal such that $\mathfrak{a} \neq R^\rho$ for any positive integer ρ, then there exists n such that $\mathfrak{a} < R^n$, for example $n=1$, therefore $\mathfrak{a} = R^n\mathfrak{b}$ for some ideal $\mathfrak{b}$. Then $\mathfrak{a} = R^n\mathfrak{b} \subseteq R^nR = R^{n+1}$, and by our assumption $\mathfrak{a}<R^{n+1}$. Thus for any integer $m\geq n$, we have $\mathfrak{a}<R^m$, therefore $\mathfrak{a}\subseteq\bigcap_{m=1}^{\infty}R^m$.

Remark. From now on, we denote $\bigcap_{n=1}^{\infty}R^n$ by $\mathfrak{d}$: $\bigcap_{n=1}^{\infty}R^n=\mathfrak{d}$.

Proposition 3. *Let R be a non-idempotent M-ring, then $R\mathfrak{d}=\mathfrak{d}R=\mathfrak{d}$.*

Proof. Since $R>R^2\supseteq\mathfrak{d}$ there exists an ideal $\mathfrak{d}'$ such that $\mathfrak{d}=R\mathfrak{d}'$, and by Theorem 2 $\mathfrak{d}'\subseteq\mathfrak{d}$ or $\mathfrak{d}'=R^k$ for some positive integer k. If $\mathfrak{d}'\subseteq\mathfrak{d}$, then $\mathfrak{d}=\mathfrak{d}'$, therefore $\mathfrak{d}=R\mathfrak{d}$; if $\mathfrak{d}'=R^k$, then $\mathfrak{d}=R\mathfrak{d}'=RR^k=R^{k+1}$, hence $\mathfrak{d}\supseteq R\mathfrak{d}=R^{k+2}\supseteq\mathfrak{d}$, therefore $\mathfrak{d}=R\mathfrak{d}$.

Proposition 4. *Let R be a non-idempotent M-ring, and let N be the Jacobson radical of R, then $N=R$ or $N\subseteq\mathfrak{d}$.*

Proof. Let $N\not\subseteq\mathfrak{d}$, then by Theorem 2 $N=R^\rho$ for some positive integer ρ. Since the Jacobson radical of $R/N=\bar{R}$ is $\{\bar{0}\}$, and $\bar{R}$ is nilpotent, it follows $\rho=1$.

Proposition 5. *Let R be a non-idempotent M-ring, $\mathfrak{a}$ any ideal contained in $\mathfrak{d}$, then $R\mathfrak{a}=\mathfrak{a}R=\mathfrak{a}$.*

Proof. Let $\mathfrak{d}>\mathfrak{a}$, then there exists ideals $\mathfrak{b},\mathfrak{b}'$ such that $\mathfrak{a}=\mathfrak{d}\mathfrak{b}=\mathfrak{b}'\mathfrak{d}$. Hence by Proposition 3 $R\mathfrak{a}=R(\mathfrak{d}\mathfrak{b})=(R\mathfrak{d})\mathfrak{b}=\mathfrak{d}\mathfrak{b}=\mathfrak{a}$.

Lemma 6. *Let R be a non-idempotent M-ring and $R^n>R^{n+1}$ for any positive integer n, then $\mathfrak{d}_1=\bigcap_{n=1}^{\infty}R^n$ is a prime ideal of R.*

Proof. If $\mathfrak{a}\mathfrak{b}\equiv 0 \pmod{\mathfrak{d}_1}$ and $\mathfrak{a}\not\equiv 0$, $\mathfrak{b}\not\equiv 0 \pmod{\mathfrak{d}_1}$ for some ideals $\mathfrak{a},\mathfrak{b}$, then by Theorem 2 $\mathfrak{a}=R^\rho$, $\mathfrak{b}=R^\nu$ for some positive integer ρ,ν, hence we have $\mathfrak{a}\mathfrak{b}=R^{\rho+\nu}\not\equiv 0 \pmod{\mathfrak{d}_1}$.

Remark. From now on, we denote the ideal denoted by $\mathfrak{d}$ by $\mathfrak{d}_1$.

Theorem 3. *Let R be a non-idempotent M-ring. We set $\mathfrak{d}_0=R$, $\mathfrak{d}_i=\bigcap_{j=1}^{\infty}\mathfrak{d}_{i-1}^j$, $i=1,2,\cdots$, and assume that there exists a positive integer n such that $\mathfrak{d}_i^m>\mathfrak{d}_i^{m+1}$ for any integer $m\geq 1$ and for any $0\leq i<n$. Then we have:*

(i) *For any ideal $\mathfrak{a}$ of R, $\mathfrak{a}\subseteq\mathfrak{d}_n$ or $\mathfrak{a}=\mathfrak{d}_j^{\rho_j}$ for some $0\leq j\leq n-1$ and positive integer ρ_j.*

(ii) *$\mathfrak{d}_1,\mathfrak{d}_2,\cdots,\mathfrak{d}_{n-1},\mathfrak{d}_n$ are prime ideals of R.*

(iii) $$\begin{aligned}&\mathfrak{d}_1=R\mathfrak{d}_1=\mathfrak{d}_1R\\&\mathfrak{d}_2=R\mathfrak{d}_2=\mathfrak{d}_2R=\mathfrak{d}_1\mathfrak{d}_2=\mathfrak{d}_2\mathfrak{d}_1\\&\quad\vdots\\&\mathfrak{d}_n=R\mathfrak{d}_n=\mathfrak{d}_nR=\mathfrak{d}_1\mathfrak{d}_n=\mathfrak{d}_n\mathfrak{d}_1=\cdots=\mathfrak{d}_{n-1}\mathfrak{d}_n=\mathfrak{d}_n\mathfrak{d}_{n-1}.\end{aligned}$$

Proof. We use an induction on n. For $n=1$, (i) follows from Theorem 2, (ii) from Lemma 6, and (iii) Proposition 3. We shall assume that the theorem holds for every integer less than n, and will prove (i), (ii), (iii) for n.

Let $\mathfrak{a}$ be an ideal such that $\mathfrak{a}\not\subseteq\mathfrak{d}_n=\bigcap_{m=1}^{\infty}\mathfrak{d}_{n-1}^m$, then $\mathfrak{a}\not\subseteq\mathfrak{d}_{n-1}^k$ for some positive integer k. Let k_0 be the minimal positive integer such that $\mathfrak{a}\not\subseteq\mathfrak{d}_{n-1}^{k_0}$. If $k_0=1$, then by the assumption of the induction we must have $\mathfrak{a}=\mathfrak{d}_j^{\rho_j}$ for $0\leq j\leq n-2$ and for some positive integer ρ_j. If $k_0>1$, then $\mathfrak{a}\subseteq\mathfrak{d}_{n-1}$, and we assume $\mathfrak{a}<\mathfrak{d}_{n-1}$. Since $\mathfrak{a}\not\subseteq\mathfrak{d}_n=\bigcap_{i=1}^{\infty}\mathfrak{d}_{n-1}^i$, we can choose the largest positive integer k such that $\mathfrak{a}\subseteq\mathfrak{d}_{n-1}^k$, then $\mathfrak{a}=\mathfrak{d}_{n-1}^k$; because if $\mathfrak{a}<\mathfrak{d}_{n-1}^k$, then $\mathfrak{a}=\mathfrak{d}_{n-1}^k\mathfrak{b}$ for some ideal $\mathfrak{b}$ such that $\mathfrak{b}\not\subseteq\mathfrak{d}_{n-1}$. Hence by the assumption of the induction $\mathfrak{b}=\mathfrak{d}_j^{\rho_j}$ for some positive integer ρ_j and j such that $0\leq j\leq n-2$. Therefore $\mathfrak{a}=\mathfrak{d}_{n-1}^k$, a contradiction.

Next we shall prove (ii). Let $\mathfrak{a}\mathfrak{b}\equiv 0 \pmod{\mathfrak{d}_n}$, $\mathfrak{a}\not\equiv 0$, $\mathfrak{b}\not\equiv 0 \pmod{\mathfrak{d}_n}$ for some ideals $\mathfrak{a}$, $\mathfrak{b}$, then by the results in (i) $\mathfrak{a}=\mathfrak{d}_{n-1}^{\rho_{n-1}}, \mathfrak{d}_{n-2}^{\rho_{n-2}}, \cdots, \mathfrak{d}_1^{\rho_1}$ or R^{ρ}, $\mathfrak{b}=\mathfrak{d}_{n-1}^{\nu_{n-1}}, \mathfrak{d}_{n-2}^{\nu_{n-2}}, \cdots, \mathfrak{d}_1^{\nu_1}$ or R^{ν}, hence $\mathfrak{a}\mathfrak{b}=\mathfrak{d}_{n-1}^{\rho_{n-1}+\nu_{n-1}}, \cdots, R^{\rho+\nu}$ contradicting the fact that $\mathfrak{a}\mathfrak{b}\equiv 0 \pmod{\mathfrak{d}_n}$.

Finally we shall prove (iii). It is sufficient to prove the fact that $\mathfrak{d}_n=R\mathfrak{d}_n=\mathfrak{d}_nR=\mathfrak{d}_1\mathfrak{d}_n=\mathfrak{d}_n\mathfrak{d}_1=\cdots=\mathfrak{d}_{n-1}\mathfrak{d}_n=\mathfrak{d}_n\mathfrak{d}_{n-1}$ only. Using the fact that R, $\mathfrak{d}_1, \cdots, \mathfrak{d}_{n-1}, \mathfrak{d}_n$ are prime ideals of R, $\mathfrak{d}_n<\mathfrak{d}_j$ $(j=0,1,\cdots,n-1)$ implies $\mathfrak{d}_n=\mathfrak{d}_j\mathfrak{a}$ for some ideal $\mathfrak{a}$, hence we have $\mathfrak{a}\equiv 0 \pmod{\mathfrak{d}_n}$ since $\mathfrak{d}_n$ is a prime ideal, and $\mathfrak{a}=\mathfrak{d}_n$, therefore $\mathfrak{d}_n=\mathfrak{d}_j\mathfrak{d}_n$.

Remark. If R is commutative, then $\mathfrak{d}_1=\{0\}$ [3; Satz 11].

Using Theorem 3 (i), we can prove the following;

Proposition 7. *Let R be a non-idempotent M-ring, then we have the series $R>R^2>\cdots>\mathfrak{d}_1>\mathfrak{d}_1^2>\cdots>\mathfrak{d}_2>\mathfrak{d}_2^2>\cdots$. We assume that in the above series we have for the first time $\mathfrak{d}_i^j=\mathfrak{d}_i^{j+1}$, then $\mathfrak{d}_{i+1}=\mathfrak{d}_{i+2}=\cdots$. If $j>1$, then $N=\mathfrak{d}_k$ for some $0\leq k\leq i$ or $N<\mathfrak{d}_{i+1}=\mathfrak{d}_{i+2}=\cdots$, and $\mathfrak{d}_{i+1}=\mathfrak{d}_{i+2}=\cdots$ is not a prime ideal of R. If $j=1$, then $N=\mathfrak{d}_k$ for some $0\leq k\leq i$ or $N<\mathfrak{d}_i=\mathfrak{d}_{i+1}=\cdots$, and $\mathfrak{d}_i=\mathfrak{d}_{i+1}=\cdots$ is a prime ideal of R. In either case, $\bigcap_{i=1}^{\infty}\mathfrak{d}_i$ is an idempotent ideal of R.*

More generally, using the transfinite induction we have the following as a generalization of Theorem 3. We denote by Λ a set of ordinals.

Theorem 4. *Let R be a non-idempotent M-ring, then we have the series:*

$$R>R^2>\cdots>R^n>R^{n+1}>\cdots>\mathfrak{d}_1,\ \mathfrak{d}_1=\bigcap_{n=1}^{\infty}R^n$$

$$\mathfrak{d}_1>\mathfrak{d}_1^2>\cdots>\mathfrak{d}_1^n>\mathfrak{d}^{n+1}>\cdots>\mathfrak{d}_2,\ \mathfrak{d}_2=\bigcap_{n=1}^{\infty}\mathfrak{d}_1^n, \cdots, \mathfrak{d}_m=\bigcap_{n=1}^{\infty}\mathfrak{d}_{m-1}^n.$$

In general, we define series $\{\mathfrak{d}_\lambda\}_\Lambda$ as follows: if α is an isolated ordinal $\mathfrak{d}_\alpha=\bigcap_{n=1}^{\infty}\mathfrak{d}_{\alpha-1}^n$, and if α is a limit ordinal $\mathfrak{d}_\alpha=\bigcap_{\beta<\alpha}\mathfrak{d}_\beta$.

Now we assume for a fixed λ, $\mathfrak{d}_\alpha^j>\mathfrak{d}_\alpha^{j+1}$ for every $\alpha<\lambda$ and every positive integer j, then we have:

(i) *Let $\mathfrak{a}$ be any ideal of R, then $\mathfrak{a}\subseteq\mathfrak{d}_\lambda$ or $\mathfrak{a}=\mathfrak{d}_\alpha^{\rho_\alpha}$ for some $\alpha<\lambda$ and some positive integer ρ_α.*

(ii) *For any $\alpha\leq\lambda$, $\mathfrak{d}_\alpha$ is a prime ideal of R.*

(iii) $\mathfrak{d}_1=R\mathfrak{d}_1=\mathfrak{d}_1R$

$$\mathfrak{d}_2=R\mathfrak{d}_2=\mathfrak{d}_2R=\mathfrak{d}_1\mathfrak{d}_2=\mathfrak{d}_2\mathfrak{d}_1$$

$$\vdots$$

$$\mathfrak{d}_\alpha=R\mathfrak{d}_\alpha=\mathfrak{d}_\alpha R=\mathfrak{d}_1\mathfrak{d}_\alpha=\mathfrak{d}_\alpha\mathfrak{d}_1=\cdots=\mathfrak{d}_\beta\mathfrak{d}_\alpha=\mathfrak{d}_\alpha\mathfrak{d}_\beta=\cdots$$

for any β, α such that $\beta<\alpha\leq\lambda$.

And as a generalization of Proposition 7:

Proposition 8. *Let R be a non-idempotent M-ring, then we have the series $\{\mathfrak{d}_\alpha\}_\Lambda$ as Theorem 4. If in the series we have for the first time $\mathfrak{d}_\lambda^j=\mathfrak{d}_\lambda^{j+1}$*[1] *for some λ and some positive integer j, then of course $\mathfrak{d}_{\lambda+1}=\mathfrak{d}_{\lambda+2}=\cdots$, and we have:*

(i) *If $j>1$, then $N=\mathfrak{d}_\beta$ for some $0\leq\beta\leq\lambda$ or $N<\mathfrak{d}_{\lambda+1}$, and $\mathfrak{d}_{\lambda+1}$ is not a prime ideal of R.*

(ii) *If $j=1$, then $N=\mathfrak{d}_\beta$ for some $0\leq\beta\leq\lambda$ or $N<\mathfrak{d}_\lambda=\mathfrak{d}_{\lambda+1}=\cdots$, and $\mathfrak{d}_\lambda=\mathfrak{d}_{\lambda+1}$ is a prime ideal of R. On either case $\mathfrak{d}=\bigcap_{\alpha\in\Lambda}\mathfrak{d}_\alpha$ is an (unique maximal) idempotent ideal of R.*

As a summary:

Theorem 5. *Let R be a non-idempotent M-ring, and $\{\mathfrak{d}_\alpha\}_\Lambda$ be the series as Theorem 4. We set $\mathfrak{d}=\bigcap_{\alpha\in\Lambda}\mathfrak{d}_\alpha$, then*

(i) *If $\mathfrak{a}$ is any ideal of R, then $\mathfrak{a}\subseteq\mathfrak{d}$ or $\mathfrak{a}=\mathfrak{d}_\beta^{\rho_\beta}$ for some $\beta<\lambda$ and some positive integer ρ_β.*

(ii) *There is a minimal $\lambda\in\Lambda$ such that $\mathfrak{d}=\mathfrak{d}_\lambda$, and for any $0\leq\alpha<\lambda$ we have $\mathfrak{d}_\alpha\mathfrak{d}=\mathfrak{d}\mathfrak{d}_\alpha=\mathfrak{d}$.*

(iii) *$\mathfrak{d}$ coincides with the unique maximal idempotent ideal of R.*[2]

Now we add some remarks:

Definition. If for every element x of a ring R, there exists a positive integer k such that $kx=0$, then we call the smallest positive integer k such that $kx=0$ *the characteristic of* R, and denote $\mathrm{ch}(R)=k$. If there is not such a k, then we set $\mathrm{ch}(R)=0$.

Let $\mathfrak{d}_i$ be any one of the series $\{\mathfrak{d}_\alpha\}_\Lambda$ in Theorem 4. Let x be any element of $\mathfrak{d}_i^j$ such that $x\notin\mathfrak{d}_i^{j+1}$, then using Theorem 4 we have $\mathfrak{d}_i^j=(RxR,\mathfrak{d}_i^{j+1})$. We define the characteristic of a element x $\mathrm{ch}(x)=k$ the smallest positive integer such that $kx\in\mathfrak{d}_i^{j+1}$: if there is not such a k, then we define $\mathrm{ch}(x)=0$.

Lemma 9. *Let x be any element of $\mathfrak{d}_i^j$ such that $x\notin\mathfrak{d}_i^{j+1}$, then* $\mathrm{ch}(x)=\mathrm{ch}(\mathfrak{d}_i^j/\mathfrak{d}_i^{j+1})$.

Proof. It follows from $\mathfrak{d}_i^j=(x,Rx,xR,RxR,\mathfrak{d}_i^{j+1})$.

Lemma 10. *Let x be any element of $\mathfrak{d}_i^j$ such that $x\notin\mathfrak{d}_i^{j+l}$, then* $\mathrm{ch}(x)$ *is a prime or zero. If $i=0$, then* $\mathrm{ch}(x)$ *is a prime.*

Proof. We assume that $\mathrm{ch}(x)$ is not zero. If $\mathrm{ch}(x)$ is not a prime

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1) We prove that $\mathfrak{d}_\lambda^j=\mathfrak{d}_\lambda^{j+1}$ actually occurs. Let $\boldsymbol{\Lambda}$ be the class of all ordinals. We set $\boldsymbol{\Lambda}_0=\{\lambda\in\boldsymbol{\Lambda}\mid\mathfrak{d}_\lambda^i\neq\mathfrak{d}_\lambda^{i+1}$ for all $i>0\}$. For every $\alpha\in\boldsymbol{\Lambda}_0$, we can choose an element x_α such that $x_\alpha\in\mathfrak{d}_\alpha$, $x_\alpha\notin\mathfrak{d}_\alpha^2$, therefore we have a one to one correspondence $\alpha\leftrightarrow x_\alpha$ between $\boldsymbol{\Lambda}_0$ and $\{x_\alpha\}\subseteq R$, so Λ_0 is a set. If we denote by $|A|$ the cardinality of a set A, then we have

$$|R|\geq|\{x_\alpha\}|=|\boldsymbol{\Lambda}_0|.$$

Therefore, if we choose a set of ordinals Λ such that $|\Lambda|>|R|$, then for some $\lambda\in\Lambda$ and some $j>0$, $\mathfrak{d}_\lambda^j=\mathfrak{d}_\lambda^{j+1}$.

2) By (i) and Proposition 8, any idempotent ideal is either contained in $\mathfrak{d}$ or is $\mathfrak{d}_\alpha^j$ for some α and some $j>0$. But the latter does not occur, therefore $\mathfrak{d}$ coincides with the unique maximal idempotent ideal of R.

and $\operatorname{ch}(x)=pq$, $p>1$, $q>1$, then by Theorem 4 $\mathfrak{d}_i=(px, Rpx, pxR, RpxR, \mathfrak{d}_i^{j+1})$, i.e. $\mathfrak{d}_i^j=(RpxR, \mathfrak{d}_i^{j+1})$, therefore for any element y of $\mathfrak{d}_i^j$ $qy \in \mathfrak{d}_i^{j+1}$ contradicting $\operatorname{ch}(x)=pq$. If $i=0$, then $R^j=(x, R^{j+1})$ where (,) means the sum of modules. It follows that $\operatorname{ch}(R^j/R^{j+1})$ is a prime.

Theorem 6. *Let R be a non-idempotent M-ring, and for $\mathfrak{d}_i \in \{\mathfrak{d}_\alpha\}_\Lambda$ let*

$$\mathfrak{d}_i > \mathfrak{d}_i^2 > \cdots > \mathfrak{d}_i^n > \mathfrak{d}_i^{n+1}$$

and suppose $\operatorname{ch}(\mathfrak{d}_i^j/\mathfrak{d}_i^{j+1}) \neq 0$, *then* $\operatorname{ch}(\mathfrak{d}_i^j/\mathfrak{d}_i^{j+1}) = \operatorname{ch}(\mathfrak{d}_i^{j+1}/\mathfrak{d}_i^{j+2}) = \cdots = \operatorname{ch}(\mathfrak{d}_i^n/\mathfrak{d}_i^{n+1}) = p_i \neq 0$ *and p_i is a prime. In case $i=0$, then for any $j \leq n$ not only $\operatorname{ch}(R^j/R^{j+1})=p_0\neq 0$ is a prime, but also the residue class ring R^j/R^{j+1} $(j\leq n)$ contains only p_0 elements.*

Proof. By Lemma 10 $\operatorname{ch}(\mathfrak{d}_i^j/\mathfrak{d}_i^{j+1})=p_i$ is a prime. Since $\mathfrak{d}_i^{j+1}>\mathfrak{d}_i^{j+2}$, we can choose elements x, y such that $x \in \mathfrak{d}_i^j$, $x \notin \mathfrak{d}_i^{j+1}=\mathfrak{d}_i^j\cdot\mathfrak{d}_i$, $y\in\mathfrak{d}_i$, $y\notin\mathfrak{d}_i^2$, and $xy\in\mathfrak{d}_i^{j+1}$, $xy\notin\mathfrak{d}_i^{j+2}$. By Lemma 9 $\operatorname{ch}(x)=p_i$, therefore $p_ix\in\mathfrak{d}_i^{j+1}$, hence $p_i\cdot xy=p_ix\cdot y\in\mathfrak{d}_i^{j+2}$. Since $\mathfrak{d}_i^{j+1}=(xy, Rxy, xyR, RxyR, \mathfrak{d}_i^{j+2})$ we can deduce $\operatorname{ch}(xy)=\operatorname{ch}(\mathfrak{d}_i^{j+1}/\mathfrak{d}_i^{j+2})\neq 0$, and therefore is a prime by Lemma 10. Therefore p_i is devisible by $\operatorname{ch}(\mathfrak{d}_i^{j+1}/\mathfrak{d}_i^{j+2})$, hence $\operatorname{ch}(\mathfrak{d}_i^{j+1}/\mathfrak{d}_i^{j+2})=p_i$. When $i=0$, the conclusion follows from $R^j=(x^j, R^{j+1})$, where x is an element of R, which does not belong to R^2.

Lemma 11. *Let $\mathfrak{o}$ be any M-ring, and let R be a non-idempotent M-ring, then the direct sum $R\oplus\mathfrak{o}$ is not a M-ring.*

Proof. We set $R^*=R\oplus\mathfrak{o}$. If R^* is a M-ring, then there exists an ideal $\mathfrak{b}$ of R^* such that $R=R^*\mathfrak{b}$, since $R<R^*$. Therefore $R=(R\oplus\mathfrak{o})\mathfrak{b}=R\mathfrak{b}\oplus\mathfrak{o}\mathfrak{b}$, hence $R\mathfrak{b}=R$ and $\mathfrak{o}\mathfrak{b}=\{0\}$. Now we denote the projection of R^* onto R by θ, and denote $\theta(\mathfrak{b})=\mathfrak{b}_1$, then $R=R\mathfrak{b}=R\mathfrak{b}_1\subseteq RR$, thus $R=R^2$, a contradiction.

Proposition 12. *Let R be a non-idempotent M-ring, then R can not be decomposed as a direct sum of ideals.*

Proof. If R is a direct sum of ideals R_1, R_2, i.e. $R=R_1\oplus R_2$, then both R_1, R_2 are M-rings. Now $R>R^2=R_1^2\oplus R_2^2$, hence $R_1^2\subseteq R_1$ and $R_2^2\subseteq R_2$, therefore $R_i^2<R_i$ for some $i=1, 2$, a contradiction.

Lemma 13. *Let R be a non-idempotent M-ring, and let $\mathfrak{a}$ be an ideal of R, then $R/\mathfrak{a}$ is a non-idempotent M-ring.*

Theorem 7. *Let R be a non-idempotent M-ring, and let R/N be completely reducible as a left R-module, then R is a radical ring, i.e. $R=N$. If furthermore R is left Noetherian, then $\mathfrak{d}_1=\{0\}$.*

Proof. Since R/N is completely reducible, R/N can not contain non-zero proper ideal by Proposition 12 and Lemma 13, hence R/N is a simple ring or a zero ring. But it can not be that $N=R^2$, therefore $N=R$. If R is left Noetherian, then by Nakayama's lemma $\mathfrak{d}_1=\{0\}$, because $N\mathfrak{d}_1=R\mathfrak{d}_1=\mathfrak{d}_1$.

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