75. On Vinogradov's Zero-Free Region for the Riemann Zeta-Function

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1. Let $\zeta(s)$ $(s=\sigma+it)$ be the Riemann zeta-function. And let $Y(t)=(\log(|t|+3))^{2/3}(\log\log(|t|+3))^{1/3}$. Also let c denote generally a positive absolute constant whose value may differ at each occurrence. Then, as is well-known, we have

Theorem 1. *$\zeta(s)$ does not vanish for $\sigma\geqq 1-cY(t)^{-1}$.*

Previous proofs of this fact are all dependent either on the theory of integral functions or on a function-theoretical lemma of Landau. The purpose of the present note is to show briefly that there exists still another proof which does not depend at all on the deep function-theoretical properties of $\zeta(s)$. Our main tools are the Vinogradov-Richert theorem (Lemma 1 below), the Selberg sieve and an argument closely related to that of [1].

As a by-product of our procedure we can prove also

Theorem 2. *Let U be sufficiently large, and let us assume*

$$\zeta(1+iU)^{-1}\ll D(U)(\log U)^{2/3}(\log\log U)^{1/4},$$

where $D(U)$ increases monotonically to infinity as $U\to\infty$. Then $\zeta(s)$ does not vanish for

$$\sigma\geqq 1-cY(U)^{-1}\log D(U),\quad |t|\leqq U/2,\quad t=\pm U.$$

The proof of Theorem 2 will not be given below; we mention only that it is derived from Lemmas 3 and 4. The detailed account will appear elsewhere.

2. Throughout in this and the next sections we assume that T is sufficiently large and that $1-\delta+iT$ is a zero of $\zeta(s)$ such that $0<\delta\leqq(\log T)^{-2/3}$. Because of the reason stated at the end of this section, we may presume also $\delta\geqq(\log T)^{-10}$.

Now let $\sigma(n;a)$ be the sum of the a-th powers of divisors of n, and let us put $f(n)=\sigma(n;-\delta-iT)$. We apply the Selberg sieve to the sequence $\{|f(n)^2|\}$. According to the general theory we should put

$$g(r)=\prod_{p|r}(F_p-1),\qquad G(R)=\sum_{r\leqq R}\mu^2(r)g(r),$$

where

$$F_p=\sum_{m=0}^{\infty}|f(p^m)|^2 p^{-m}.$$

Then the optimal weight is given by

$$\theta_d = G(R)^{-1}\mu(d)\Big\{\sum_{\substack{r \leqq R|d \\ (r,d)=1}} \mu^2(r)g(r)\Big\}\prod_{p|d} F_p. \tag{1}$$

Lemma 1 ([3]). *Uniformly for* $0 \leqq \sigma \leqq 1$ *and* $|t| \geqq 2$ *we have*

$$\zeta(s) \ll |t|^{c(1-\sigma)^{3/2}} (\log|t|)^{2/3}.$$

Lemma 2. *Let* θ_d *be defined by* (1), *and let us put*

$$S(N) = \sum_{n \leqq N} |f(n)|^2 \Big(\sum_{d|n} \theta_d\Big)^2.$$

Then we have, provided $R \geqq \exp(B\,Y(T))$ *and* $N \geqq R^4$,

$$S(N) \ll_B \delta N.$$

Lemma 3. *Let* U *be sufficiently large, and let* $N \geqq \exp(B\,Y(U))$. *Then we have*

$$\sum_{n \leqq N} |\sigma(n\,;iU)|^4 \ll_B N(\log N)^5 |\zeta(1+iU)|^8 (\log U)^{4/3}.$$

Lemma 4 (A. Selberg). *Let* $z > 1$ *be arbitrary. And let* $\lambda_d = \mu(d)$ *if* $d \leqq z$, $= \mu(d)(\log z^2/d)/(\log z)$ *if* $z < d \leqq z^2$, *and* $= 0$ *otherwise. Further let* $\xi > 1$ *be such that* $(\xi - 1)^{-1} = O(\log z)$. *Then we have*

$$\sum_{n=1}^{\infty} \Big(\sum_{d|n} \lambda_d\Big)^2 n^{-\xi} = O(1).$$

To prove Lemma 2 we should observe that for $\sigma > 1$

$$\sum_{n=1}^{\infty} |f(n)|^2 n^{-s} = \zeta(s)\zeta(s+2\delta)\zeta(s+\delta+iT)\zeta(s+\delta-iT)\zeta(2(s+\delta))^{-1}. \tag{2}$$

From this and Lemma 1 we get easily, on the conditions given above,

$$S(N) \ll_B \zeta(1+2\delta)\,|\zeta(1+\delta+iT)|^2\, G(R)^{-1}N.$$

On the other hand, just as Lemma 8 of [2], Lemma 1 and (2) give also, for $R \geqq \exp(B\,Y(T))$,

$$G(R) \gg_B \delta^{-1}\zeta(1+2\delta)\,|\zeta(1+\delta+iT)|^2.$$

Hence the assertion of Lemma 2 follows. As for Lemma 3 we note that for $\sigma > 1$

$$\begin{aligned}\sum_{n=1}^{\infty} |\sigma(n\,;iU)|^4 n^{-s} \\ = \zeta(s)^6\zeta(s+iU)^4\zeta(s-iU)^4\zeta(s+2iU)\zeta(s-2iU)K(s),\end{aligned} \tag{3}$$

where $K(s)$ is regular and bounded for $\sigma > 1/2$. This and Lemma 1 give rise to Lemma 3. (3) should be compared with the famous

$$\zeta(\sigma)^3\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geqq 1 \qquad (\sigma > 1).$$

As is shown in [4, pp. 43–44] this inequality and some elementary estimates of $\zeta(s)$ alone yield that $\zeta(s)^{-1} = O((\log(|t|+2))^7)$ for $\sigma \geqq 1 - c(\log(|t|+2))^{-9}$. And this is sufficient for the proof of Lemma 4, as can be seen from the proof of Lemma 5 of [2] which is a generalization of Lemma 4. This remark is essential, for in the present note we are not allowed to use the theory of integral functions and the lemma of of Landau.

3. Now we give a brief proof of Theorem 1. Let ω_d be defined by

$$\sum_{d|n} \omega_d = \Big(\sum_{d|n} \theta_d\Big)\Big(\sum_{d|n} \lambda_d\Big). \tag{4}$$

We put

$$M(s)=\sum_{d\leq z^2R}\omega_d d^{-s}\prod_{p|d}\left(1+\frac{1}{p^{\delta+iT}}-\frac{1}{p^{s+\delta+iT}}\right).$$

Then we have, for $\sigma>1$,

$$\zeta(s)\zeta(s+\delta+iT)M(s)=1+\sum_{z\leq n}f(n)\left(\sum_{d|n}\omega_d\right)n^{-s}. \tag{5}$$

Here we set, with a certain large constant A,

$$z=\exp(4A\ Y(T)),\quad R=\exp(A\ Y(T)),\quad X=\exp(10A\ Y(T)).$$

Next we consider the Mellin integral

$$\frac{1}{2\pi i}\int_{(\sigma=\eta)}\zeta(s)\zeta(s+\delta+iT)M(s)X^{s-\rho}\Gamma(s-\rho)ds.$$

where $\rho=1-\delta+iT$, and $\eta=1-(\log\log T)^{2/3}(\log T)^{-2/3}$. By Lemma 1 and (5) we find

$$1\ll\sum_{z\leq n}f(n)\left(\sum_{d|n}\omega_d\right)n^{-\rho}e^{-n/X}.$$

Thus by (4) we get

$$1\ll\sum_{z\leq n\leq X^2}|f(n)|^2\left(\sum_{d|n}\theta_d\right)^2 n^{-1}\sum_{n\leq X^2}\left(\sum_{d|n}\lambda_d\right)^2 n^{-1+2\delta}.$$

From this and Lemmas 2 and 4 we infer

$$1\ll\delta Y(T)X^{4\delta},$$

which obviously ends the proof of Theorem 1.

References

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