63. On Equations Defining Abelian Varieties and Modular Functions

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To a pair (z, e) of a point z of the Siegel upper half space $\mathcal{H}_n$ of degree n and an n-square matrix e with coefficients in $\boldsymbol{Z}$ and $\det e \neq 0$, we can attach a complex torus $\boldsymbol{C}/\langle z, e\rangle$ with the period lattice subgroup $\langle z, e\rangle$ of $\boldsymbol{C}^n$ generated by the column vectors of (z, e). For a vector $k=\begin{pmatrix} k' \\ k'' \end{pmatrix} \in \boldsymbol{R}^{2n}$, the theta function $\vartheta[k](z\,|\,x)$ is defined by

$$\vartheta[k](z\,|\,x)=\sum_{r\in\boldsymbol{Z}^n} \boldsymbol{e}\Big(\frac{1}{2}{}^t(r+k')z(r+k')+{}^t(r+k')(x+k'')\Big),$$

which is a holomorphic function of $(z, x) \in \mathcal{H}_n \times \boldsymbol{C}^n$. Given $\beta \in \boldsymbol{Z}_+$, $U(\beta^t e)$ denotes a complete set of representatives of $\beta^{-1}{}^t e^{-1}\boldsymbol{Z}^n \bmod \boldsymbol{Z}^n$. A map $\varphi^{(z)}$ from $\boldsymbol{C}^n$ to the projective space $\boldsymbol{P}\beta^{n|\det e|-1}(\boldsymbol{C})$ defined by $x \mapsto \Big(\cdots, \vartheta\begin{bmatrix} k' \\ 0 \end{bmatrix}(z\,|\,x), \cdots\Big)_{k' \in U(\beta^t e)}$ induces a projective embedding of $\boldsymbol{C}^n/\langle z, e\rangle$ if $\beta \geqslant 3$. The Im $(\varphi^{(z)})$ is an abelian variety, which is denoted by $A(z)$. The purpose of this note is to write down explicitly a system of equations defining $A(z)$ and to show that the set of quotients of coefficients of the equations generates the field of modular functions with respect to the principal congruence subgroup $\Gamma_{{}^t e}(\beta)$.

We shall indicate some definitions and notations. For a commutative ring R having the unity 1, $M(n\times\alpha, R)$ (or $M(n, R)$, resp.) is the set of $(n\times\alpha)$-matrices (or n-square matrices) with coefficients in R; in particular, $M(n\times 1, R)$ is denoted by R^n. For a matrix $e \in M(n, \boldsymbol{Z})$ with $\det e \neq 0$, the paramodular group $\Gamma_{{}^t e}$ or the principal congruence subgroup $\Gamma_{{}^t e}(\beta)$ of level β, $\beta \in \boldsymbol{Z}_+$, is defined, respectively, by $\Gamma_{{}^t e} = \Big\{M \in M(2n, \boldsymbol{Z})\,|\,{}^t M\begin{pmatrix} 0 & -{}^t e \\ e & 0 \end{pmatrix}M=\begin{pmatrix} 0 & -{}^t e \\ e & 0 \end{pmatrix}\Big\}$ or $\Gamma_{{}^t e}(\beta)=\Big\{M \in \Gamma_{{}^t e}\,|\,M = 1_{2n}+M'\begin{pmatrix} e & 0 \\ 0 & {}^t e \end{pmatrix},\ M' \in M(2n, \boldsymbol{Z})\Big\}$. When we write $k=\begin{pmatrix} k' \\ k'' \end{pmatrix} \in \boldsymbol{R}^{2n}$, k' are k'' are upper and lower halves of k in $\boldsymbol{R}^n$. For e (or β) as above, $U(e)$ (or $U(\beta)$) is a complete set of representatives of $e^{-1}\boldsymbol{Z}^n$ (or $\beta^{-1}\boldsymbol{Z}^n$) modulo $\boldsymbol{Z}^n$. On the other hand, the residue group $e^{-1}\boldsymbol{Z}^n/\boldsymbol{Z}^n$ and its character group are denoted, respectively, by $\tilde{U}(e)$ and $\tilde{U}^*(e)$. We put $\boldsymbol{e}(t)=\exp(2\pi\sqrt{-1}t)$ for $t \in \boldsymbol{C}$.

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More precise details and proofs will be discussed in a separate paper.

1. Theta relations of higher degree. Let α be an integer $\geqslant 2$ and let T be a matrix in $M(\alpha, \boldsymbol{Z})$ defined by

$$T=\begin{bmatrix} 1 & \alpha-1 & 0 & \cdots & 0 & 0 \\ 1 & -1 & \alpha-2 & \cdots & 0 & 0 \\ 1 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 2 & 0 \\ 1 & -1 & -1 & \cdots & -1 & 1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \end{bmatrix}. \tag{1.1}$$

If we write ${}^tT^{-1}=T^*$, we have

(1.1′)

$$T^*={}^tT^{-1}=\begin{bmatrix} \alpha^{-1} & \alpha^{-1} & 0 & \cdots & 0 & 0 \\ \alpha^{-1} & -\alpha^{-1}(\alpha-1)^{-1} & (\alpha-1)^{-1} & \cdots & 0 & 0 \\ \alpha^{-1} & -\alpha^{-1}(\alpha-1)^{-1} & -(\alpha-1)^{-1}(\alpha-2)^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & 3^{-1} & 0 \\ \alpha^{-1} & -\alpha^{-1}(\alpha-1)^{-1} & -(\alpha-1)^{-1}(\alpha-2)^{-1} & \cdots & -6^{-1} & 2^{-1} \\ \alpha^{-1} & -\alpha^{-1}(\alpha-1)^{-1} & -(\alpha-1)^{-1}(\alpha-2)^{-1} & \cdots & -6^{-1} & -2^{-1} \end{bmatrix}.$$

For $\alpha-1$ vectors $l_i=\begin{pmatrix} l'_i \\ l''_i \end{pmatrix} \in \boldsymbol{R}^{2n}$, a holomorphic function of $(z, y_1, \cdots, y_{\alpha-1}) \in \mathcal{H}_n \oplus \left(\overset{\alpha-1}{\oplus} \boldsymbol{C}^n\right)$ $\check{\Xi}(l_1, \cdots, l_{\alpha-1}|z|y_1, \cdots, y_{\alpha-1})$ is defined by

$$\begin{aligned} \check{\Xi}(l_1, \cdots, l_{\alpha-1}|z|y_1, \cdots, y_{\alpha-1}) &= \sum_{\substack{p_i \in U(i) \\ i=\alpha-1,\cdots,2}} \vartheta\begin{bmatrix} l'_1-p_{\alpha-1} \\ l''_1 \end{bmatrix}(\alpha(\alpha-1)z|y_1) \\ &\times \vartheta\begin{bmatrix} l'_2+p_{\alpha-1}-p_{\alpha-2} \\ l''_2 \end{bmatrix}((\alpha-1)(\alpha-2)z|y_2) \cdots \vartheta\begin{bmatrix} l'_{\alpha-2}-p_3+p_2 \\ l''_{\alpha-2} \end{bmatrix}(6z|y_{\alpha-2}) \\ &\times \vartheta\begin{bmatrix} l'_{\alpha-1}+p_2 \\ l''_{\alpha-1} \end{bmatrix}(2z|y_{\alpha-1}). \end{aligned} \tag{1.2}$$

Then we have (Appendix in [2])

$$\begin{aligned} \prod_{i=1}^{\alpha} \vartheta\begin{bmatrix} k'_i \\ k''_i \end{bmatrix}(z|x_i) \\ = \sum_{p \in U(\alpha)} \vartheta\begin{bmatrix} l'_0+p \\ l''_0 \end{bmatrix}(\alpha z|y) \check{\Xi}\left(l_1+\begin{pmatrix} p \\ 0 \end{pmatrix}, l_2, \cdots, l_{\alpha-1}|z|y_1, \cdots, y_{\alpha-1}\right), \end{aligned} \tag{1.3}$$

where $(l_0 l_1 \cdots l_{\alpha-1})=\begin{pmatrix} l'_0 l'_1 \cdots l'_{\alpha-1} \\ l''_0 l''_1 \cdots l''_{\alpha-1} \end{pmatrix}=\begin{pmatrix} (k'_1 k'_2 \cdots k'_\alpha)T^* \\ (k''_1 k''_2 \cdots k''_\alpha)T \end{pmatrix}$ and $(y y_1 \cdots y_{\alpha-1}) =(x_1 x_2 \cdots x_\alpha)T$.

Let χ be a character in $\tilde{U}^*(\alpha)$. We put

$$\begin{aligned} \check{\Xi}(\chi|l_1, \cdots, l_{\alpha-1}|z|y_1, \cdots, y_{\alpha-1}) \\ = \sum_{p \in U(\alpha)} \chi(-p) \check{\Xi}\left(l_1+\begin{pmatrix} p \\ 0 \end{pmatrix}, l_2, \cdots, l_{\alpha-1}|z|y_1, \cdots, y_{\alpha-1}\right). \end{aligned} \tag{1.4}$$

Then, under the same notations as in (1.3) we have

$$\sum_{p\in U(\alpha)}\chi(p)\prod_{i=1}^{\alpha}\vartheta\left[k_i+\binom{p}{0}\right](z\,|\,x_i)$$

(1.5)

$$=\left(\sum_{p\in U(\alpha)}\chi(p)\vartheta\left[l_0+\binom{p}{0}\right](\alpha z\,|\,y)\right)\check{\Xi}(\chi\,|\,l_1,\cdots,l_{\alpha-1}\,|\,z\,|\,y_1,\cdots,y_{\alpha-1}).$$

Let $(k_{01},k_{02},\cdots,k_{0\alpha})$ and $(k_{11},k_{12},\cdots,k_{1\alpha})$ be two systems of vectors in $\boldsymbol{R}^{2n}$. As in (1.3), we put

$$(l_{j0}l_{j1}\cdots l_{j(\alpha-1)})=\begin{pmatrix}l'_{j0}l'_{j1}\cdots l'_{j(\alpha-1)}\\ l''_{j0}l''_{j1}\cdots l''_{j(\alpha-1)}\end{pmatrix}=\begin{pmatrix}(k'_{j0}k'_{j1}\cdots k'_{j\alpha})T^*\\ k''_{j0}k''_{j1}\cdots k''_{j\alpha})T\end{pmatrix},\qquad (j=0,1),$$

and

$$(yy_1\cdots y_{\alpha-1})=(x_1x_2\cdots x_\alpha)T.$$

If $l'_{00}\equiv l'_{10}\ (\mathrm{mod}\ \alpha^{-1}\boldsymbol{Z}^n)$ and $l''_{00}=l''_{10}$, then we have

$$\check{\Xi}(\chi\,|\,l_{11},\cdots,l_{1(\alpha-1)}\,|\,z\,|\,y_1,\cdots,y_{\alpha-1})\left(\sum_{p\in U(\alpha)}\chi(p)\prod_{i=1}^{\alpha}\vartheta\left[k_{0i}+\binom{p}{0}\right](z\,|\,x_i)\right)$$

(1.6)

$$=\chi(l'_{10}-l'_{00})\check{\Xi}(\chi\,|\,l_{01},\cdots,l_{0(\alpha-1)}\,|\,z\,|\,y_1,\cdots,y_{\alpha-1})$$

$$\times\left(\sum_{p\in U(\alpha)}\chi(p)\prod_{i=1}^{\alpha}\vartheta\left[k_{1i}+\binom{p}{0}\right](z\,|\,x_i)\right).$$

In particular we consider this formula (1.6) in the case where $k''_{j1}=k''_{j2}=\cdots=k''_{j\alpha}=0$, $j=0,1$, and $x_1=x_2=\cdots=x_\alpha=x$. Under this assumption, we have $l''_{j0}=\cdots=l''_{j(\alpha-1)}=0$ and $y_1=\cdots=y_{\alpha-1}=0$. Thus we put

(1.7) $$\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,z)=\check{\Xi}\left(\chi\,\middle|\,\binom{l'_1}{0},\cdots,\binom{l'_{\alpha-1}}{0}\,\middle|\,z\,\middle|\,0,\cdots,0\right)$$

for $\chi\in\tilde{U}^*(\alpha)$ and $(l'_1,\cdots,l'_{\alpha-1})\in\boldsymbol{M}(n\times(\alpha-1),\boldsymbol{R})$.

After suitable substitutions we have a formula containing $\Xi(\chi\,|\,l'_{j1},\cdots,l'_{j(\alpha-1)}\,|\,z)$ instead of $\check{\Xi}(\chi\,|\,l_{j1},\cdots,l_{j(\alpha-1)}\,|\,z\,|\,y_1,\cdots,y_{\alpha-1})$, $(j=0,1)$, which is a special case of the formula (1.6).

2. Equations defining the abelian variety $A(z)$. Now, besides $e\in\boldsymbol{M}(n,\boldsymbol{Z})$ with $\det e\neq 0$ we fix two integers $\beta\geqslant 3$ and $\alpha>1$ such that β is divisible by α. We also fix a complete set $U(\beta^t e)$ of representatives once for all. Let $\{X(\tilde{k}')\,|\,k'\in U(\beta^t e)\}$ be a set of independent indeterminates, bijectively corresponding to $\tilde{U}(\beta^t e)$, where $\tilde{k}'$ is the congruence class determined by $k'\in U(\beta^t e)$. $(\cdots,X(\tilde{k}'),\cdots)$ can be considered as the coordinate variables of the ambient projective space of $A(z)$, which is the projective embedding of the complex torus $\boldsymbol{C}^n/\langle z,e\rangle$ by $x\mapsto\left(\cdots,\vartheta\begin{bmatrix}k'\\0\end{bmatrix}(\beta z\,|\,\beta x),\cdots\right)$.

For $\tilde{l}'$ or $(\tilde{l}',\chi)$, $\tilde{l}'\in\tilde{U}(\beta^t e)$ and $\chi\in\tilde{U}^*(\alpha)$, a set $L(\tilde{l}')$ of indices or a set $F(\chi,\tilde{l}')$ of holomorphic functions on $\mathcal{H}_n$ are defined respectively by

(2.1) $$L(\tilde{l}')=\left\{(k'_1,\cdots,k'_\alpha)\,|\,k'_i\in U(\beta^t e),\ i=1,\cdots,\alpha;\ \sum_{i=1}^{\alpha}\tilde{k}'_i=\tilde{l}'\right\},$$

and

$$F(\chi,\tilde{l}')=\{\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,\beta z)\,|\,(l'_1\cdots l'_{\alpha-1})=(k'_1\cdots k'_\alpha)T^*_1,\ (\tilde{k}'_1,\cdots,\tilde{k}'_\alpha)\in L(\tilde{l}')\},\tag{2.2}$$

where T^*_1 is the $\alpha\times(\alpha-1)$-matrix obtained from T^* by excluding the first column.

$(k'_{j1},\cdots,k'_{j\alpha})$ and $(l'_{j0},\cdots,l'_{j(\alpha-1)})$ being as in (1.6), for a triple $(\chi,(\tilde{k}'_{0i})_{1\leqslant i\leqslant\alpha},(\tilde{k}'_{1i})_{1\leqslant i\leqslant\alpha})$ such that (k'_{0i}) and (k'_{1i}) belong to the same $L(\tilde{l}')$ for some $\tilde{l}'\in\tilde{U}(\beta^t e)$, we define an α-form (α-homogeneous polynomial) in variables $\{X(\tilde{k})\}_{\tilde{k}\in\tilde{U}(\beta^t e)}$:

$$\begin{aligned}P(\chi\,|\,(\tilde{k}'_{0i}),(\tilde{k}'_{1i})\,|\,X(\tilde{k}'))=\;&\Xi(\chi\,|\,l'_{11},\cdots,l'_{1(\alpha-1)}\,|\,\beta z)\sum_{p\in U(\alpha)}\chi(p)\prod_{i=1}^{\alpha}X(\widetilde{k'_{0i}+p})\\&-\chi(l'_{10}-l'_{00})\Xi(\chi\,|\,l'_{01},\cdots,l'_{0((\alpha-1)}\,|\,\beta z)\\&\times\sum_{p\in U(\alpha)}\chi(p)\prod_{i=1}^{\alpha}X(\widetilde{k'_{1i}+p}).\end{aligned}\tag{2.3}$$

Furthermore, we define a set $I(z)$ of polynomials by

$$I(z)=\{P(\chi\,|\,\tilde{k}'_{0i}),(\tilde{k}'_{1i})\,|\,X(\tilde{k}'))\,|\,\chi\in\tilde{U}^*(\alpha),(k'_{0i})\text{ and }(k'_{1i})\in L(\tilde{l}'),\ \tilde{l}'\in\tilde{U}(\beta^t e)\},\tag{2.4}$$

and for a point $z_0\in\mathcal{H}_n$, $I(z_0)$ is the set of α-forms in $C[\{X(\tilde{k}')\}_{\tilde{k}'\in\tilde{U}(\beta^t e)}]$ obtained from $I(z)$ through substituting z by z_0.

Then our theorem is formulated in the following way:

Theorem. (0) *If we substitute* $(X(\tilde{k}'))$ *by* $\left(\vartheta\begin{bmatrix}k'\\0\end{bmatrix}(\beta z\,|\,\beta x)\right)$ *in* $P(\chi\,|\,(\tilde{k}'_{0i}),(\tilde{k}'_{1i})\,|\,X(\tilde{k}'))$, *it vanishes identically as a function of* $(z,x)\in\mathcal{H}_n\times C^n$.

(I) *For an arbitrary* $(\chi,\tilde{l}',z_0)\in\tilde{U}^*(\alpha)\times\tilde{U}(\beta^t e)\times\mathcal{H}_n$, *there is a function* $\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,\beta z)$ *in* $F(\chi,\tilde{l}')$, *which does not vanish at* z_0.

(II) *Given* $z_0\in\mathcal{H}_n$, *the space of α-forms in* $C[\{X(\tilde{k}')\}_{\tilde{k}'\in\tilde{U}(\beta^t e)}]$ *vanishing on* $A(z_0)$ *is spanned by* $I(z_0)$.

(III) *The abelian variety* $A(z_0)$ *is the common zero set of* $I(z_0)$.

(IV) *The quotient of two functions in an* $F(\chi,\tilde{l}')$, *whose denominator does not identically vanish, is a modular function with respect to* $\Gamma_{te}(\beta)$.

(V) *The field of modular functions with respect to* $\Gamma_{te}(\beta)$ *is generated by*

$$\bigcup_{(\chi,l')\in\tilde{U}^*(\alpha)\oplus U(\beta^t e)}\{\Xi(\chi\,|\,l'_{01},\cdots,l'_{0(\alpha-1)}\,|\,\beta z)/\Xi(\chi\,|\,l'_{11},\cdots,l'_{1(\alpha-1)}\,|\,\beta z)\,|\,\Xi(\chi\,|\,l'_{01},\cdots\,|\,\beta z)\text{ and }\Xi(\chi\,|\,l'_{11},\cdots\,|\,\beta z)\in F(\chi,\tilde{l}'),\ \Xi(\chi\,|\,l'_{11},\cdots\,|\,\beta z)\neq 0\}.$$

The assertions (0)–(III) are known in the case $\beta=4$ and $\alpha=2$ ([1], [3]). In the case $\beta=9$ and $\alpha=3$, the assertions (0)–(II) are proved in [5].

3. The Fourier expansion of $\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,z)$. We can determine series expansions of the functions $\check{\Xi}(l_1,\cdots,l_{\alpha-1}\,|\,z\,|\,y_1,\cdots,y_{\alpha-1})$ etc. explicitly, when $(l_1,\cdots,l_{\alpha-1})$ and $(y_1,\cdots,y_{\alpha-1})$ are given as in (1.3). Here we restrict ourselves to considering only the function $\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,z)$.

Let α, ζ, K and O be, respectively, a prime number, a primitive α-th root of the unity, the field $\boldsymbol{Q}(\zeta)$ and the ring $\boldsymbol{Z}[\zeta]$. $\mathrm{Tr}\,M$ means the trace of the matrix M, and on the other hand $\mathrm{tr}\,(\lambda)$, $\lambda=\begin{pmatrix}\lambda_{(1)}\\ \vdots \\ \lambda_{(n)}\end{pmatrix}\in K^n$, means the vector in $\boldsymbol{Q}^n$ whose coefficients are given by the trace of the element $\lambda_{(i)}$ in K, over $\boldsymbol{Q}$.

Given $\chi\in\tilde{U}^*(\alpha)$ and $\lambda\in K^n$, we define a holomorphic function $\xi(\chi,\lambda,z)$ on $\mathfrak{H}_n$ by

$$(3.1)\qquad \xi(\chi,\lambda,z)=\sum_{\sigma\in\mathfrak{O}^n}\chi(\alpha^{-1}\,\mathrm{tr}\,\sigma)\boldsymbol{e}\Big(\frac{1}{2}\,\mathrm{Tr}\,((\mathrm{tr}\,((\lambda+\sigma)_{(i)}\overline{(\lambda+\sigma)}_{(j)}))_{(i,j)}z)\Big),$$

where $(\mathrm{tr}\,((\lambda+\sigma)_{(i)}\overline{(\lambda+\sigma)}_{(j)}))_{(i,j)}$ is a positive semi-definite symmetric matrix in $M(n,\boldsymbol{Q})$.

For $(k'_1,\cdots,k'_\alpha)\in M(n\times\alpha,\boldsymbol{Q})$, if we put $\lambda=k'_1+k'_2\zeta+\cdots+k'_\alpha\zeta^{\alpha-1}$ and $(l'_1,\cdots,l'_{\alpha-1})$ as in (1.3), then we have

$$(3.2)\qquad \xi(\chi,\lambda,\alpha^{-1}z)=\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,z).$$

Using $\xi\Big(\chi,\lambda,\dfrac{\beta}{\alpha}z\Big)$ instead of $\Xi(\chi\,|\,l'_1,\cdots,l'_{\alpha-1}\,|\,\beta z)$ we can formulate Theorem in the previous section. In this case the set $F(\chi,\tilde{l}')$ in (2.2) is replaced by

$$(3.3)\qquad \mathcal{F}(\chi,\tilde{l}')=\Big\{\xi\Big(\chi,\lambda\frac{\beta}{\alpha}z\Big)\,\Big|\,\lambda\in\mathcal{U}(\beta^t e),\quad \mathrm{tr}\,\lambda'\equiv -l'\,\mathrm{mod}\,\alpha\beta^{-1t}e^{-1}\boldsymbol{Z}^n\Big\},$$

where $\mathcal{U}(\beta^t e)$ is a complete set of representatives of $\beta^{-1t}e^{-1}\mathfrak{O}^n\ \mathrm{mod}\ \mathfrak{O}^n$.

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