49. *Examples of Obstructed Holomorphic Maps*

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1. By a complex space, we mean a reduced, Hausdorff, complex analytic space. For a complex manifold X, the set $S(X)$ of all compact complex submanifolds of X has a complex space structure (Douady [1]). For $V \in S(X)$, $\dim T_V S(X) \leq \dim H^0(V, \mathcal{O}(N))$, where $T_V S(X)$ is the Zariski tangent space to $S(X)$ at V and $\mathcal{O}(N)$ is the sheaf of sections of the normal bundle N along V (see Namba [4]). We say that V is *unobstructed relative to* X if $S(X)$ is non-singular at V and $\dim T_V S(X) = \dim H^0(V, \mathcal{O}(N))$. Otherwise, V is said to be *obstructed relative to* X. If $H^1(V, \mathcal{O}(N)) = 0$, then V is unobstructed relative to X (Kodaira [2]). Examples of obstructed submanifolds were given by Zappa [5] and Mumford [3].

Now, let V and W be compact complex manifolds. Then, the set $\mathrm{Hol}(V, W)$ of all holomorphic maps of V into W is a complex space. In fact, by identifying $f \in \mathrm{Hol}(V, W)$ with the graph $\Gamma_f \subset V \times W$, $\mathrm{Hol}(V, W)$ is regarded as an open subspace of $S(V \times W)$. Note that the normal bundle along Γ_f is canonically isomorphic to the pull back f^*TW over f of the tangent bundle TW of W. We say that $f \in \mathrm{Hol}(V, W)$ is *unobstructed* if the graph Γ_f is unobstructed relative to $V \times W$. Otherwise, f is said to be *obstructed.* It seems that no example of obstructed holomorphic maps is known. The purpose of this note is to give such examples.

2. Let V be a compact Riemann surface of genus g. Let $\boldsymbol{P}^1$ be the complex projective line. Then $\mathrm{Hol}(V, \boldsymbol{P}^1)$ is nothing but the set of all meromorphic functions on V. It is divided into open (and closed) subspaces:

$$\mathrm{Hol}(V, \boldsymbol{P}^1) = \mathrm{Const} \cup R_1(V) \cup R_2(V) \cup \cdots,$$

where Const is the set of all constant functions and $R_n(V)$ is the set of all meromorphic functions on V of (mapping) order n. If $n \geq g+1$, then $R_n(V)$ is non-empty. If $n \geq g$, then every $f \in R_n(V)$ is unobstructed so that $R_n(V)$ is non-singular and of dimension $2n+1-g$. This follows from the fact that $f^*T\boldsymbol{P}^1 = [2D_\infty(f)]$, where $D_\infty(f)$ is the polar divisor of f and $[2D_\infty(f)]$ is the line bundle determined by the divisor $2D_\infty(f)$.

Theorem. *For* $f \in R_{g-1}(V)$, *assume that* $[2D_\infty(f)] = K_V$ (*the canonical bundle of* V). *Then f is unobstructed if and only if*

$$2\dim|D_\infty(f)|+1=g.$$

Proof. If f is unobstructed, then there is a connected non-singular open neighbourhood U of f in $R_{g-1}(V)$ such that every $h\in U$ is also unobstructed. In particular, $\dim H^0(V,\mathcal{O}([2D_\infty(h)]))=g$. Hence

(1) $$[2D_\infty(h)]=K_V, \qquad \text{for all } h\in U.$$

Let $J(V)$ be the Jacobi variety of V. Consider the following holomorphic maps:

$$\alpha: h\in U\mapsto[D_\infty(h)-(g-1)P_0]\in J(V),$$
$$\beta: x\in J(V)\mapsto 2x\in J(V),$$

where $P_0\in V$ is the base point. The map β is an (unramified) covering map of order 2^{2g}. (1) implies that the composition $\beta\alpha$ is constant on U. Hence α must be constant, i.e.,

$$[D_\infty(h)]=[D_\infty(f)], \qquad \text{for all } h\in U.$$

This means that h is obtained as a pencil in $|D_\infty(f)|$. Hence

$$g=\dim U=2\dim|D_\infty(f)|+1.$$

The converse is easy. Q.E.D.

By Clliford's theorem, we have

Corollary. *Let V be a non-hyperelliptic compact Riemann surface of genus $g\geq 4$. For $f\in R_{g-1}(V)$, assume that $[2D_\infty(f)]=K_V$. Then f is obstructed.*

3. Let V be a non-hyperelliptic compact Riemann surface of genus 4. The canonical curve C of V is the complete intersection of a quadric surface F and a cubic surface G in $\boldsymbol{P}^3$ meeting transversally. Assume that F is singular, i.e., a quadric cone. Then the ruling on F gives $f\in R_3(V)$ which satisfies $[2D_\infty(f)]=K_V$. Hence, f is obstructed. In this case, $R_3(V)\cong \mathrm{Aut}(\boldsymbol{P}^1)$ (the automorphism group of $\boldsymbol{P}^1$), so that $\dim R_3(V)=3$, while $\dim H^0(V,\mathcal{O}([2D_\infty(f)]))=4$.

For example, let V be the normalization of the closure in $\boldsymbol{P}^2$ of the curve: $y^3=x^6-1$. Then, the meromorphic function $f=x$ on V is obstructed.

4. Let $V=C$ be a non-singular plane quintic curve. It is non-hyperelliptic and has the genus 6. The line sections on C determine a complete linear system $|D|$ such that $[2D]=K_C$. Hence the projection π_P with the center $P\in\boldsymbol{P}^2-C$ is obstructed. Moreover, we can show that

(1) every element of $R_5(C)$ is obtained in this way, hence is obstructed,

(2) $R_5(C)$ is a principal $\mathrm{Aut}(\boldsymbol{P}^1)$-bundle over $\boldsymbol{P}^2-C$.

5. Let V be the normalization of the closure in $\boldsymbol{P}^2$ of the curve: $y^3=x^8-1$. Then V is non-hyperelliptic and has the genus 7. The meromorphic function $f=x^2$ on V satisfies $[2D_\infty(f)]=K_V$. Hence, f is obstructed. Moreover, we can show that

(1) $R_6(V)$ is singular at $f=x^2$: the tangent cone at f is given by

$\{(z_1, z_2, \cdots, z_7) \in \boldsymbol{C}^7 \mid z_1 z_2 = 0\}$,

(2) $R_3(V) \cong \mathrm{Aut}(\boldsymbol{P}^1)$,

(3) $R_4(V)$ and $R_5(V)$ are empty.

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