36. Asymptotic Distribution of Eigenvalues of a Class of Hypoelliptic Operators*)

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Through the fundamental solution of a heat (or parabolic) equation, fractional powers of the Laplacian Δ (or elliptic operator) have been studied by many authors (e.g. see [2], [4], and [5]). However for hypoelliptic operators no work similar to the above has appeared yet in the literature. This note is to announce some results of a similar treatment for hypoelliptic operators. Full details will appear in a separate publication.

1. We call a C^∞-function $\lambda(x,\xi)$ on $R^n_x \times R^n_\xi$ *a basic weight function* when it satisfies the following conditions:

$$\text{(i)}\quad A^{-1}(1+|x|+|\xi|)^a \le \lambda(x,\xi) \le A(1+|x|^{\tau_0}+|\xi|) \qquad (a\ge 0,\ \tau_0\ge 0,\ A>0)$$

$$\text{(1.1)}\quad \text{(ii)}\quad |\lambda^{(\alpha)}_{(\beta)}(x,\xi)| \le A_{\alpha,\beta}\lambda(x,\xi)^{1-|\alpha|+\delta|\beta|} \qquad \text{for any } \alpha,\beta \quad (-\infty<\delta<1,\ A_{\alpha,\beta}>0)$$

$$\text{(iii)}\quad \lambda(x+y,\xi) \le A_1(1+|y|)^{\tau_1}\lambda(x,\xi) \qquad (\tau_1\ge 0,\ A_1>0)$$

where α and β are multi-indices of non-negative integers α_j and β_j with $|\alpha|=\alpha_1+\cdots+\alpha_n$, $|\beta|=\beta_1+\cdots+\beta_n$; $\lambda^{(\alpha)}_{(\beta)}(x,\xi)=(\partial/\partial\xi)^\alpha(-i\,\partial/\partial x)^\beta\lambda(x,\xi)$.

By $S^m_{\lambda,\rho,\delta}$ we denote the set of all functions (or symbols) in $C^\infty(R^n_x\times R^n_\xi)$ satisfying

$$\text{(1.2)}\qquad |p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta}\lambda(x,\xi)^{m-\rho|\alpha|+\delta|\beta|} \qquad \text{for any } \alpha,\beta \quad (-\infty<m<\infty,\ 0\le\rho\le 1,\ -\infty<\delta<1,\ \delta<\rho).$$

For any $p(x,\xi)\in S^m_{\lambda,\rho,\delta}$ the corresponding pseudo differential operator $P(x,D)$ of $p(x,\xi)$ is defined by

$$P(x,D)u(x)=\int_{R^n_\xi} e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi$$

where u belongs to the class of rapidly decreasing functions of Schwartz,

$$\hat{u}(\xi)=\int_{R^n_x} e^{ix\cdot\xi}u(x)dx \quad\text{and}\quad d\xi=(2\pi)^{-n}d\xi.$$

If a polynomial $p(x,\xi)=\sum_{|\alpha|\le m}a_\alpha(x)\xi^\alpha$ in $S^m_{\lambda,\rho,\delta}$ satisfies

$$\text{(1.3)}\qquad p_0(x,\xi)=\operatorname{Re}p(x,\xi)\ge C_0\lambda(x,\xi)^m, \qquad \text{for } |x|+|\xi|\ge R$$

then it can be shown that the differential operator

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$$P(x,D)=\sum_{|\alpha|\leq m} a_\alpha(x)(-i\partial/\partial x)^\alpha$$

is hypoelliptic by means of the existence of a left parametric of $P(x,D)$ (see § 3 below).

2. We consider a equation of parabolic type with initial condition

$$(2.1)\qquad \begin{aligned} &Lu=\partial u/\partial t+P(x,D)u=0 \qquad \text{in } (0,\infty)\times R^n_x \\ &u|_{t=0}=u_0 \end{aligned}$$

and call $E(t)$ a *fundamental solution* of (2.1) if $LE(t)=0$ in $t>0$ and $E(0)=I$.

Theorem 1. *For any $p(x,\xi)\in S^m_{\lambda,\rho,\delta}$ satisfying* (1.3), *we can construct a fundamental solution $E(t)=e(t;x,D)$ (as a pseudo differential operator of* (2.1) *such that for any N with $-N(\rho-\delta)+m\leq 0$*

$$e(t;x,\xi)=\sum_{j=0}^{N-1}\tilde{e}(t,x,\xi)+f_N(t;x,\xi)$$

where

$$\operatorname{supp} e(t;x,\xi)\subseteq\{\xi\in R^n_\xi : |\xi|\geq l\}$$

$$\tilde{e}_0(t;x,\xi)=\exp(-tp(x,\xi))$$

$$\tilde{e}_j(t;x,\xi)\exp(-\varepsilon tp_0(x,\xi))\in S^{-(\rho-\delta)j}_{\lambda,\rho,\delta}$$

for any fixed $\varepsilon>0$

$$|f^{(\alpha)}_{N(\beta)}(t;x,\xi)|\leq C\lambda(x,\xi)^{m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)N}\exp(-C(1+|x|+|\xi|)^{am}).$$

Moreover $E(t)$ is unique in a class of operators in $L^2(R^n_x)$.

$\tilde{e}_j(t;x,\xi)$ $(j\geq 1)$ are obtained by truncating $e_j=e_j(t;x,\xi)$ as zero in $\{\xi:|\xi|\leq l\}$, where e_j are successively defined by

$$(2.2)\qquad \begin{aligned} &\{\partial/\partial t+p(x,\xi)\}e_j=-q_j \\ &q_j=\sum_{k=0}^{j-1}\sum_{|\alpha|+k=j}p^{(\alpha)}(x,\xi)e_{k(\alpha)}(t;x,\xi)/\alpha! \end{aligned}$$

Let $\tilde{F}_N(t)$ be the operator, defined by its symbol $\sigma(\tilde{F}_N(t))=\sum_{j=0}^N\tilde{e}_j(t;x,\xi)$, which is a right parametrix of L in (2.1):

$$L\tilde{F}_N(t)=\tilde{R}_N(t),\qquad t>0$$

where $\sigma(\tilde{R}_N(t))\cdot\exp((1-\varepsilon)p_0(x,\xi))\in S^{m(\rho-\delta)N}_{\lambda,\rho,\delta}$ for any N. Using $\tilde{F}_N(t)$ we can construct $E(t)$ by E. E. Levi's method by means of symbol calculus (see [8]).

3. In what follows we assume further that P is a formally self-adjoint strictly positive operator whose extension in $L^2(R^n_x)$ (with domain $C_0^\infty(R^n_x)$) is denoted by $\tilde{P}$. Let $\mathring{S}^m_{\lambda,\rho,\delta}$ be the subclass of $S^m_{\lambda,\rho,\delta}$ for which $C_{\alpha,\beta}=C_{\alpha,\beta}(x)$ in (1.2) tends to zero as $|x|\to\infty$. A $p(x,\xi)$ of $S^m_{\lambda,\rho,\delta}$ is called slowly varying in $S^m_{\lambda,\rho,\delta}$ if $p_{(\beta)}(x,\xi)\in\mathring{S}^{m+\delta|\beta|}_{\lambda,\rho,\delta}$ for any $\beta\neq 0$. (For $\lambda=|\xi|$ see [1] and [3]).

Theorem 2. *$\tilde{P}^{-1}$ is a completely continuous operator from $L^2(R^n_x)$ to $L^2(R^n_x)$.*

The condition (1.3) assures that $\tilde{P}$ possesses a right parametrix Q_N with $\tilde{P}Q_N=I+T_N$, where $Q_N\in\mathring{S}^{-m}_{\lambda,\rho,\delta}$ and $T_N\in\mathring{S}^{-(\rho-\delta)N}_{\lambda,\rho,\delta}$ by (1.1). Combining the complete continuity of Q_N and T_N with the boundedness of $\tilde{P}^{-1}$ in $L^2(R^n_x)$, the theorem will be proved by using the equation $\tilde{P}^{-1}=Q_N-\tilde{P}^{-1}T_N$.

4. $\tilde{P}$ has a unique spectral measure $\mu=\mu(\lambda)$ whose support is contained in $[\lambda_0+\infty)$ with $\tilde{P}=\int_{\lambda_0}^{+\infty} d\mu$. The uniqueness of the fundamental solution $E(t)$ implies that

(4.1) $$E(t)=\int_{\lambda_0}^{+\infty} e^{-\lambda t}d\mu(\lambda) \qquad (t>0).$$

By Theorem 2 and the hypoellipticity of $\tilde{P}$ it can be shown that $\mu(\lambda)$ has a spectral function $\mu(\lambda; x, y)$ (which is continuous in x and y under suitable conditions, cf. [7]):

$$\mu(\lambda; x, y)=\sum_{\lambda_j\leq\lambda} \phi_j(x)\overline{\phi_j(y)},$$

where $\{\lambda_j\}_{j=0}^{\infty}$ is a sequence of discrete eigenvalues of $\tilde{P}$ and $\{\phi_j\}_{j=0}^{\infty}$, a complete orthonormal system of eigenfunctions ϕ_j corresponding to λ_j of $\tilde{P}$ in $L^2(R_x^n)$. Thus $e(t; x, y)$ obtained in § 2 may be written as

(4.2) $$e(t; x, \xi)=e^{-ix\cdot\xi} \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x)\overline{\hat{\phi}_j(\xi)} \qquad (t>0).$$

We now define *fractional powers of* $\tilde{P}$ with complex parameter z as follows

$$\tilde{P}^z=\int_{\lambda_0}^{+\infty} \lambda^z d\mu(\lambda)$$

which may also be written as

$$\tilde{P}^z=\sum_{j=0}^{\infty} \lambda_j^z(\cdot, \phi_j)_{L^2}\phi_j.$$

Using the result of Theorem 1, we can define

(4.3) $$p_z(x, \xi)=\frac{1}{\Gamma(-z)}\int_0^{+\infty} t^{-z-1}e(t; x, \xi)dt, \qquad \operatorname{Re} z<0$$

where $\Gamma(-z)$ is the gamma function. From (4.2) and (4.3) it follows

(4.4) $$p_z(x, \xi)=e^{-ix\cdot\xi} \sum_{j=0}^{\infty} \lambda_y^z\phi_j(x)\overline{\hat{\phi}_j(\xi)}$$

provided that $\sum_{j=0}^{\infty} \lambda_j^{\operatorname{Re} z}<\infty$.

Theorem 3. (1) *For* $\operatorname{Re} z<0$, $p_z(x, \xi)$ *is analytic in* z.

(2) $$p_z(x, \xi)\sim\sum_{j=0}^{\infty} p_{zj}(x, \xi)$$

where

$$p_{zj}(x, \xi)=\frac{1}{\Gamma(-z)}\int_0^{\infty} t^{-z-1}\tilde{e}_j(t; x, \xi)dt.$$

Moreover $p_z(x, \xi)-\sum_{j=0}^{N} p_{zj}(x, \xi)$ *is analytic in* z *for* $\operatorname{Re} z<0$.

5. We shall apply Theorem 3 to obtain an estimate of asymptotic distribution of eigenvalues of $\tilde{P}$ under the additional condition:

(5.1) $p_0(x, \xi)$ is a polynomial in $(x, \xi) \in R_{x,\xi}^{2n}$.

By a use of the integral in (4.3) we define *the* ζ*-function* $\zeta(z)$ of $P(x, D)$ as follows

(5.2) $$\zeta(z)=\int_{R_\xi^n} \left\{\int_{R_x^n} p_z(x, \xi)dx\right\}d\xi$$

which is absolutely convergent for $\operatorname{Re} z<-2n/am$.

Adopting the idea of Smagin [6], we have

Theorem 4. (1) $\zeta(z)$ *is analytic in the half-plane* $\operatorname{Re} z<-2n/am$. (2) $\zeta(z)$ *can be continuously extended to the entire complex plane as* *a meromorphic function with poles of multiplicity not greater than 2n in a finite number of real arithmetic progressions.* (3) *The first pole of the Laurent expansion of the function $\zeta(z)$ coincides with the first pole of the Laurant expansion of the integral*

$$\zeta_0(z)=\int_{R^n_\xi}\left\{\int_{R^n_x}\left(\frac{1}{\Gamma(-z)}\int_0^{+\infty} t^{-z-1}e_0(t\,;x,\xi)dt\right)dx\right\}d\xi. \tag{5.3}$$

6. From (4.4) and (5.2) follows

$$\begin{aligned}\zeta(z)&=\int_{R^n_\xi}\left\{\int_{R^n_x} p_z(x,\xi)dx\right\}d\xi\\ &=\sum_{j=0}^{\infty}\lambda_j^z\int_{R^n_\xi}\left\{\int_{R^n_x}\phi_j(x)e^{-ix\cdot\xi}dx\right\}\overline{\hat{\phi}_j(\xi)}d\xi\\ &=\sum_{j=0}^{\infty}\lambda_j^z\end{aligned}$$

which implies that

$$\zeta(z)=\int_1^\infty t^z dN(t)$$

where $N(t)=\sum_{\lambda_j\le t}1$.

Finally, let r be the first pole of $\zeta_0(z)$ in (5.3) and K (an integer) its degree. Then

$$\zeta(z)(z-r)^K\to A\neq 0 \qquad \text{as } z\to r.$$

Hence by Ikehara's tauberian theorem we get

$$\lim_{t\to\infty}\frac{N(t)}{t^r(\ln t)^K}=\frac{(-1)^{K-1}A}{(K-1)\,!}.$$

We thus have

Theorem 5. *Given* (1.1), (1.3), *and* (5.1) *the following asymptotic formula for $N(t)$ of $\tilde{P}$ holds as $t\to\infty$:*

$$N(t)=O(t^r(\ln t)^K)$$

where r is the first pole of $\zeta_0(z)$ and K is its multiplicity.

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