# 14. On Commutative Rings which have Completely Reducible Torsion Theories

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Throughout this note we assume that $R$ is a commutative ring with identity and all $R$-modules are unital. Let $(T,F)$ be a torsion theory for $R$-mod. We call that $T(F)$ is completely reducible if every $R$-module belonging to $T(F)$ is completely reducible, and call $(T,F)$ completely reducible when both $T$ and $F$ are completely reducible.

The purpose of this paper is to study completely reducible torsion theories. It is shown in Theorem 2.5 that if $R$ has a completely reducible torsion theory, then it is a (von Neumann) regular ring whose spectrum of prime ideals has only a finite number of non-isolated points and in this case all completely reducible torsion theories are determined by a finite subset of the spectrum containing all non-isolated points.

A torsion theory is a pair $(T,F)$ of subclasses of $R$-mod satisfying

(1) $T\cap F=\{0\}$,

(2) $T$ is closed under homomorphic images,

(3) $F$ is closed under submodules, and

(4) for each $A$ in $R$-mod, there is a submodule $T(A)$ of $A$ called the torsion submodule of $A$ such that $T(A)\in T$ and $A/T(A)\in F$.

$T$ is then called the torsion class and $F$ is called the torsion-free class.

A torsion theory $(T,F)$ is called hereditary when $T$ is closed under submodules.

Let $(T,F)$ be a torsion theory. $T(F)$ is said to be a $TTF$-class if there is a subclass $U\subseteq R$-mod for which $(U,T)$ $((F,U))$ forms a torsion theory. $F$ is a $TTF$-class iff $F$ is closed under homomorphic images.

**1.** We denote the Boolean ring consisting of all idempotents in $R$ by $B(R)$ and the spectrum of prime ideals of $B(R)$ by $X(R)$. $X(R)$ forms a Boolean space with the family $\{U(e)\mid e\in B(R)\}$ as an open basis, where $U(e)=\{x\in X(R)\mid e\in x\}$. For an $R$-module $A$ and $x$ in $X(R)$, we set $Ax=\{ae\mid a\in A,\ e\in x\}$. Remark that all factor rings $R/Rx$ for $x$ in $X(R)$ are indecomposable as a ring, and that $x$ in $X(R)$ is an isolated point iff $x=B(R)(1-e)$ for some minimal idempotent $e$ in $R$.

We need the following lemmas for the later use, but we omit the proofs.

**Lemma 1.1.** *Let $X$ be a non-empty closed subset of $X(R)$.*

(a) *If $Y$ is a subset of $X(R)$ such that $X \subseteq Y$ and $\bigcap_{x \in X} Rx = \bigcap_{y \in Y} Ry$, then $X = Y$.*

(b) $\bigcap_{x \in X} Ax = A\left(\bigcap_{x \in X} x\right)$ *for any $R$-module $A$.*

**Lemma 1.2.** *Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be a set of minimal idempotents of $R$ and set $X = X(R) - \{B(R)(1-e_\lambda) \mid \lambda \in \Lambda\}$. Then $\sum_{\lambda \in \Lambda} \mathrm{Re}_\lambda = \bigcap_{x \in X} Rx$.*

**2.** For a non-empty subset $X$ of $X(R)$, we define

$T_X = \{A \in R\text{-mod} \mid A = Ax \text{ for all } x \text{ in } X\}$,

$F_X = \{B \in R\text{-mod} \mid \mathrm{Hom}_R(A, B) = 0 \text{ for all } A \text{ in } T_X\}$.

Then, as is easily seen, $(T_X, F_X)$ forms a hereditary torsion theory with $T_X(A) = \bigcap_{x \in X} Ax$ for all $A \in R$-mod (cf. [2]). In case $X$ is empty, we simply define as $T_X = R$-mod and $F_X = \{0\}$.

**Theorem 2.1.** *The following conditions are equivalent for a subset $X$ of $X(R)$:*

(a) *$X$ is closed.*

(b) *$F_X$ is a TTF-class.*

(c) *$F_X = T_{X^c}$, where $X^c = X(R) - X$.*

*Therefore, symmetrically, the following conditions are also equivalent:*

(a′) *$X$ is open.*

(b′) *$T_X$ is a TTF-class.*

(c′) *$T_X = F_{X^c}$.*

**Proof.** (a)$\Rightarrow$(b). Since $X$ is closed, it follows from (b) of Lemma 1.1 that $T_X(R)$ is idempotent, $T_X = \{A \in R\text{-mod} \mid A = AT_X(R)\}$ and $F_X = \{A \in R\text{-mod} \mid AT_X(R) = 0\}$. Hence, by [3, Theorem 2.1], $F_X$ is a *TTF*-class.

(b)$\Rightarrow$(a). Assume that $X$ is not closed and let $y \in X^- - X$, where $X^-$ denotes the closure of $X$ in $X(R)$. Since $(R/Ry)x = (Rx + Ry)/Ry = R/Ry$ for all $x$ in $X$, we see $R/Ry \in T_X$. Hence $(R/Ry)T_X(R) = R/Ry$ and so $R = T_X(R) + Ry$. Let $1 = s + re$, where $s \in T_X(R)$, $r \in R$ and $e \in y$. For any $x$ in $U(e) \cap X$, we have $re \in x$ and $s \in x$ and hence $1 \in x$. This is false. Thus $X$ must be closed.

(c)$\Rightarrow$(b). Clear.

(a), (b)$\Rightarrow$(c). If $A \in T_{X^c}$, then $A = Ay$ for all $y \in X^c$, which implies that $T_X(A) = \bigcap_{x \in X} Ax = \bigcap_{x \in X(R)} Ax$ and hence $T_X(A) = 0$ by [4, Proposition 1.7]. Thus $A \in F_X$ and $T_{X^c} \subseteq F_X$. Conversely let $A \in F_X$ and $y \in X^c$. Choose $e$ in $B(R)$ so that $y \in U(e)$ and $U(e) \cap X = \emptyset$. Then $1 - e \in \bigcap_{x \in X} x \subseteq T_X(R)$ from which we get $A(1-e) = 0$. This implies that $A = Ay$ for all $y \in X^c$, that is, $A \in T_{X^c}$. Consequently $F_X \subseteq T_{X^c}$.

We denote the Goldie torsion theory by $(G, N)$. $N$ is the class of all non-singular $R$-modules and $G$ is the smallest torsion class containing all singular $R$-modules ([1]).

**Corollary 2.2.** *If $R$ is a semiprime ring with essential socle, then $(G, N)=(T_X, F_X)$, where $X$ denotes the (open) subset of isolated points in $X(R)$.*

**Proof.** Denote the socle of $R$ by $S$. By (b) of Lemma 1.1, $\bigcap_{x \in X^c} Ax = A\left(\bigcap_{x \in X^c} x\right)$ and, by Theorem 2.1, we have $F_{X^c}=T_X$. Since $R$ is semiprime, $S$ is generated by the set of minimal idempotents of $R$. Lemma 1.2 shows that $S=\bigcap_{x \in X^c} Rx$. Thus the result follows from the examination of $A \in G \Leftrightarrow AS=0 \Leftrightarrow A\left(\bigcap_{x \in X^c} x\right)=0 \Leftrightarrow A \in F_{X^c}=T_X$.

**Proposition 2.3.** *For a non-empty subset of $X(R)$, the following conditions are equivalent:*

(a) *$F_X$ is completely reducible.*

(b) *$R/T_X(R)$ is a direct sum of fields.*

(c) *$X$ is a finite set such that, for each $x$ in $X$, $R/Rx$ is a field.*

**Proof.** (a)$\Rightarrow$(b). This is evident.

(b)$\Rightarrow$(c). Since $R/T_X(R)$ is artinian, clearly $T_X(R)=\bigcap_{x \in X'} Rx$ for some finite subset $X'$ of $X$. But $X'=X$ by (a) of Lemma 1.1. Putting $X=\{x_1, \cdots, x_n\}$, clearly $R/T_X(R) \simeq R/Rx_1 \oplus \cdots \oplus R/Rx_n$. Hence it follows that each $R/Rx_i$ is a field.

(c)$\Rightarrow$(a). Again put $X=\{x_1, \cdots, x_n\}$. $X$ is then closed and $R/T_X(R) \simeq R/Rx_1 \oplus \cdots \oplus R/Rx_n$. Therefore we can see from Theorem 2.1 that $F_X$ is completely reducible.

**Proposition 2.4.** *The following conditions are equivalent for a given torsion theory $(T, F)$.*

(a) *$T$ is completely reducible and $F$ is a $TTF$-class.*

(b) *$(T, F)=(T_X, F_X)$ for some $X \subseteq X(R)$ such that each point in $X^c$ is an isolated point and each $Rx$ for $x$ in $X^c$ is a maximal ideal of $R$.*

**Proof.** (a)$\Rightarrow$(b). Since $F$ is a $TTF$-class, by [3, Theorem 2.1], $T(R)$ is idempotent, $T=\{A \in R\text{-mod} \mid AT(R)=A\}$ and $F=\{A \in R\text{-mod} \mid AT(R)=0\}$. Since $T(R)$ is idempotent and completely reducible, $T(R)=\sum_{\lambda \in \Lambda} \mathrm{Re}_\lambda$ for some set $\{e_\lambda \mid \lambda \in \Lambda\}$ of minimal idempotents in $R$. Note that all $B(R)(1-e_\lambda)$ for $\lambda \in \Lambda$ are isolated points of $X(R)$ and all $R(1-e_\lambda)$ for $\lambda \in \Lambda$ are maximal ideals of $R$. Setting $X=X(R)-\{B(R)(1-e_\lambda) \mid \lambda \in \Lambda\}$, $X$ is closed and $T(R)=\sum_{\lambda \in \Lambda} \mathrm{Re}_\lambda=\bigcap_{x \in X} Rx=T_X(R)$ by Lemma 1.2. Thus, by Theorem 2.1, we have $(T, F)=(T_X, F_X)$.

(b)$\Rightarrow$(a). Since $X$ is closed, by Theorem 2.1, $F_X$ is a $TTF$-class. Since all points of $X^c$ are isolated points, $X^c=\{B(R)(1-e_\lambda) \mid \lambda \in \Lambda\}$ for some set $\{e_\lambda \mid \lambda \in \Lambda\}$ of minimal idempotents in $R$. Lemma 1.2 shows that $T_X(R)=\bigcap_{x \in X} Rx=\sum_{\lambda \in \Lambda} Re_\lambda$ and then each $Re_\lambda$ is a minimal ideal of $R$.

Thus $T_X(R)$ is completely reducible and so is each member of $T$ by Theorem 2.1.

We are now in a position to show our main result.

**Theorem 2.5.** *If $R$ has a completely reducible torsion theory, then it is a regular ring such that $X(R)$ has only a finite number of non-isolated points. In this case, for any finite subset $X$ of $X(R)$ containing all non-isolated points, $(T_X, F_X)$ is completely reducible and all completely reducible torsion theories are obtained in this way.*

**Proof.** Let $(T, F)$ be a completely reducible torsion theory. Then, by Propositions 2.3 and 2.4, $(T, F)=(T_X, F_X)$ for some finite subset $X \subseteq X(R)$ such that all points in $X^c$ are isolated and all $Rx$ for $x$ in $X^c$ are maximal ideals. For $x$ in $X$, $Rx$ is also a maximal ideal, since $R/Rx$ is a member of $F_X$ and is indecomposable as a ring. Thus we conclude that $R$ is a regular ring (see [4, pp. 40–41]). For any finite subset $X$ of $X(R)$ containing all non-isolated points, $(T_X, F_X)$ is indeed completely reducible by Propositions 2.3 and 2.4.

The space obtained by one point compactification of any infinite discrete space is a Boolean space which has only one non-isolated point.

## References

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