15. On Gallagher's Prime Number Theorem

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1. Summarising our recent investigations [3] [4] [5], we show here very briefly a quite simple proof of Gallagher's prime number theorem [2; Theorem 7] *without* appealing to the zero-density theorem of Linnik type or to the Deuring-Heilbronn phenomenon. Our argument can be considered to be a penetration of Selberg's sieve into one of the deepest areas of the theory of prime numbers. To state the theorem we use the following convention: If there is an exceptional zero $1-\delta$ of $L(s,\chi_1)$, χ_1 real primitive (mod q_1), such that $\delta\leqq(\log Q)^{-1}$, $q_1\leqq Q$, then we put

$$\tilde{\psi}(x,\chi)=\sum_{n<x}\chi(n)\Lambda(n)(1+\chi_1(n)n^{-\delta}).$$

Otherwise we delete the factor $1+\chi_1(n)n^{-\delta}$. Then a slight modification of Gallagher's theorem states that

Theorem. *There exist effective constants $A_0, c_1, c_2>0$ such that*

$$\sum_{1<q\leqq Q}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)}|\tilde{\psi}(x+h,\chi)-\tilde{\psi}(x,\chi)|\leqq c_1\,\mathrm{Min}\,(1,\delta\log x)he^{-c_2A},$$

whenever $Q^A\leqq x/Q\leqq h\leqq x$, $A_0\leqq A$.

As is easily seen, this implies Fogels' prime number theorem [1] and thus Linnik's theorem [6; Kap. X]. Our estimations below are very rough, and in all probabilities a detailed study of our argument will provide A_0, c_1, c_2 with fairly good explicit values.

2. We may restrict ourselves to the case in which an exceptional zero does exist. Otherwise the argument of [3] can be used. In what follows $B(n)$, $g(n)$, $G(R)$, $\Psi_r(n)$, $\Phi_r(s)$ are all defined in [4]. Also we assume always that r is square-free and that ε is a sufficientry small positive constant. Constants implied by the Vinogradov and the Landau symbols are all effective.

Lemma 1. *Let*

$$G_q(R)=\sum_{\substack{r\leqq R\\(r,q)=1}}g(r),\qquad K(q)=\prod_{p|q}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{\chi_1(p)}{p^{1+\delta}}\right)^{-1}.$$

Then we have $G_q(R)\geqq K(q)^{-1}G(R)$.

Lemma 2. *Let $c(n)$ be arbitrary complex numbers. Then we have, for any $0<N\leqq M$,*

$$\sum_{\substack{q\leqq Q\\r\leqq R\\(q,r)=1}}K(q)g(r)\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)}\left|\sum_{M<n\leqq M+N}\chi(n)\Psi_r(n)B(n)^{1/2}c(n)\right|^2$$
$$\leqq\{L(1+\delta,\chi_1)N+O((M^{1/2}R^3Q^5)^{1+\varepsilon})\}\sum_{M<n\leqq M+N}|c(n)|^2.$$

Lemma 3. *If* $\sum n|c(n)|^2$ *converges, then we have, for any* $T\geqq 1$,

$$\sum_{\substack{q\leqq Q\\ r\leqq R\\ (q,r)=1}} K(q)g(r) \sum_{\chi(\mathrm{mod}\ q)}^{*} \int_{-T}^{T} \left|\sum_{n=1}^{\infty} \chi(n)\Psi_r(n)B(n)^{1/2}c(n)n^{it}\right|^2 dt$$

$$\ll \sum_{n=1}^{\infty} \{L(1+\delta,\chi_1)n + T(n^{1/2}R^3Q^5)^{1+\varepsilon}\}|c(n)|^2.$$

Lemma 4. *Let* $F(s,\chi)=L(s,\chi)L(s+\delta,\chi\chi_1)$, $\chi\ (\mathrm{mod}\ q)$ *non-principal. Then we have, for* $\mathrm{Re}(s)\geqq 1-c(\log qq_1|s|)^{-1}$,

$$\frac{F'}{F}(s,\chi)\ll \log(qq_1|s|).$$

To prove Lemma 2 we consider the dual form

$$\sum_{M<n\leqq M+N} B(n)\left|\sum_{\substack{q\leqq Q\\ r\leqq R\\ (r,q)=1}} (K(q)g(r))^{1/2} \sum_{\chi(\mathrm{mod}\ q)}^{*} \chi(n)\Psi_r(n)b(\chi,r)\right|^2,$$

where $b(\chi,r)$ are arbitrary complex numbers. Expanding out, we encounter sums of the sort

$$S(\chi,\chi';r,r')=\sum_{M<n\leqq M+N} \chi\bar{\chi}'\Psi_r\Psi_{r'}(n)B(n),$$

where $\chi\bar{\chi}'\Psi_r\Psi_{r'}(n)=\chi(n)\chi'(n)\Psi_r(n)\Psi_{r'}(n)$ and $\chi\ (\mathrm{mod}\ q)$, $\chi'\ (\mathrm{mod}\ q')$, $(q,r)=(q',r')=1$. So we are led to the function

$$\sum_{n=1}^{\infty} \chi\bar{\chi}'\Psi_r\Psi_{r'}(n)B(n)n^{-s}=L(s,\chi\bar{\chi}')L(s+\delta,\chi\bar{\chi}'\chi_1)A_{r,r'}(s,\chi\bar{\chi}'),$$

where the explicit form $A_{r,r'}(s,\chi\bar{\chi}')$ can easily be obtained by expressing the left side in an Euler product. And we see that the residue of the right side at $s=1$ is $E(\chi,\chi';r,r')L(1+\delta,\chi_1)(K(q)g(r))^{-1}$ where $E(\chi,\chi';r,r')$ is 1 if $(\chi,r)=(\chi',r')$, and $=0$ otherwise. So we have, by the routine complex integration method,

$$S(\chi,\chi';r,r')=E(\chi,\chi';r,r')L(1+\delta,\chi_1)(K(q)g(r))^{-1}N+O((M^{1/2}R^2Q^3)^{1+\varepsilon}).$$

This gives the assertion of the lemma. Then Lemma 3 can be immediately obtained by Gallagher's mean value theorem [2; Theorem 1]. Lemmas 1 and 4 are easy.

3. Now we put

$$U_r(s,\chi)=(1-F(s,\chi)\Phi_r(s,\chi)g(r)^{-1})^2,$$

where in the definition [4; Lemma 3] of $\Phi_r(s,\chi)$ we use ξ_d of [4; Lemma 4]. Then by the familiar argument we have, for non-principal $\chi\ (\mathrm{mod}\ q)$, $q\leqq Q$,

$$\tilde{\psi}(x+h,\chi)-\tilde{\psi}(x,\chi)=-\frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} \frac{F'}{F}(s,\chi)U_r(s,\chi)((x+h)^s-x^s)s^{-1}ds$$
$$+O\{x^{1/2}(r^2zQ^2T)^{1+\varepsilon}+T^{-1}(r^2x)^{1+\varepsilon}\},$$

where $\eta=1-c(\log QT)^{-1}$. Multiplying by $K(q)g(r)$ both sides and summing over $r\leqq Q$, χ primitive $(\mathrm{mod}\ q)$, $q\leqq Q$, $(q,r)=1$, we have

$$G(Q)\sum_{q\leqq Q}\sum_{\chi(\mathrm{mod}\ q)}^{*}|\tilde{\psi}(x+h,\chi)-\tilde{\psi}(x,\chi)|$$

$$\ll h(\log QT)\exp\left(-c\frac{\log x}{\log QT}\right)I(\eta)+x^{1/2}(zQ^6T)^{1+\varepsilon}+T^{-1}(xQ^4)^{1+\varepsilon},$$

where we have used Lemmas 1 and 4, and also we have put

$$I(\sigma)=\sum_{\substack{q,r\leqq Q\\(q,r)=1}} K(q)g(r) \sum_{\chi(\mathrm{mod}\ q)}^{*} \int_{\sigma-iT}^{\sigma+iT} |U_r(s,\chi)||ds|.$$

Then we have, by [4; Lemma 3] and Lemma 3 above,

$$I(\kappa)\ll \sum_{n\geqq z} B(n)\left(\sum_{d|n}\xi_d\right)^2 \{L(1+\delta, x_1)n+T(n^{1/2}Q^7)^{1+\varepsilon}\}n^{-2\kappa},$$

where $\kappa=1+(\log QT)^{-1}$. So, if we put $z=(T^2Q^{15})^{1+2\varepsilon}$, we get $I(\kappa)\ll L(1+\delta,\chi_1)$, since we have [4; Lemma 4] and $B(n)\leqq\tau(n)$. On the other hand we see easily that $I\left(\frac{1}{2}\right)\ll(zQ^6T^2)^{1+\varepsilon}$. Hence by the convexity argument [6; p. 404] we find $I(\eta)\ll L(1+\delta,\chi_1)$. That is, we have, by the second assertion of [5; Lemma 4],

$$\sum_{q\leqq Q}\sum_{\chi(\mathrm{mod}\ q)}^{*} |\tilde{\psi}(x+h,\chi)-\tilde{\psi}(x,\chi)|$$
$$\ll h\delta\,(\log QT)\exp\left(-c\frac{\log x}{\log QT}\right)+(x^{1/2}Q^{21}T^3)^{1+4\varepsilon}+T^{-1}(xQ^4)^{1+\varepsilon}.$$

And taking $T=Q^6x^{\varepsilon}$, we end our brief proof of the theorem.

References

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