8. *On the Deuring-Heilbronn Phenomenon. II*

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1. Quite recently two simple proofs of the Deuring-Heilbronn phenomenon [4] have been obtained independently by the present author [6] and Jutila [2]. Jutila's proof can be much simplified by appealing to the weight $\Psi_r(n)$ of [6]. But, compared with [2], the real advantage of [6] is in its Lemma 4. To exhibit this, we prove here very briefly a hybrid of two fundamental theorems of Linnik [3] [4] coupled with further simplifications which are embodied in Lemmas 2 and 3 below and which show that whole things are now reduced to a simple application of the Selberg sieve. Similar simplifications are, of course, applicable to the former proofs of Linnik's zero-density theorem [3]. Our new result is as follows:

Theorem. *Let* $1-\delta$ *be the exceptional zero of* $L(s,\chi_1)$, χ_1 *real* (mod q). *And let* $\tilde{N}(\alpha,T,\chi)$ *denote the number of zeros of* $L(s,\chi)L(s+\delta,\chi\chi_1)$ *in the region* $\mathrm{Re}\,(s)\geqq\alpha$, $|\mathrm{Im}\,(s)|\leqq T$. *Then we have, for* $\alpha>3/4$,

$$\sum_{\chi\,(\mathrm{mod}\,q)}\tilde{N}(\alpha,T,\chi)\ll_\varepsilon\delta(\log qT)(q^7T^4)^{(1+\varepsilon)((1-\alpha)/(3\alpha-2))}.$$

This may not be the best exponent attainable by our method. A similar but much weaker result can be found in [1; Théorème 14], which was obtained by the power-sum method of Turán. The large sieve extension can be proved quite similarly.

2. In what follows, $B(n)$, $g(r)$, $G(R)$ are all defined in [6].

Lemma 1. *Let*

$$(f^{(1)}\circ f^{(2)})_d=\sum_{[u,v]=d}f_u^{(1)}f_v^{(2)}.$$

Then we have

$$\sum_{d|n}(f^{(1)}\circ f^{(2)})_d=\Big(\sum_{u|n}f_u^{(1)}\Big)\Big(\sum_{v|n}f_v^{(2)}\Big).$$

Lemma 2. *Let* $\eta_d=O(|\mu(d)|\,d^\varepsilon)$ *and let*

$$F(s,\chi;\eta)=\sum_{d=1}^{\infty}\chi(d)d^{-s}\eta_d\prod_{p|d}\left(1+\frac{\chi_1(p)}{p^\delta}-\frac{\chi\chi_1(p)}{p^{1+\delta}}\right).$$

Then we have, for $\mathrm{Re}\,(s)>1$,

$$\sum_{n=1}^{\infty}\chi(n)B(n)\Big(\sum_{d|n}\eta_d\Big)n^{-s}=L(s,\chi)L(s+\delta,\chi\chi_1)F(s,\chi;\eta).$$

Lemma 3. *Let*

$$G_d(R)=\sum_{\substack{r\leqq R\\(r,d)=1}}\mu^2(r)g(r),$$

and let

$$\theta_d=\mu(d)\prod_{p|d}\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{\chi_1(p)}{p^{1+\delta}}\right)G_{dq}(R/d)G_q(R)^{-1}.$$

Then we have $|\theta_d|\leqq|\mu(d)|$ *and*

$$\sum_{\substack{d,f\leqq R\\(df,q)=1}}\frac{\theta_d\theta_f}{[d,f]}\prod_{\substack{p|d\\p|f}}\left(1+\frac{\chi_1(p)}{p^\delta}-\frac{\chi_1(p)}{p^{1+\delta}}\right)=G_q(R)^{-1}.$$

Lemma 4. *We have*
$G_q(R)\geqq q^{-1}\varphi(q)G(R),\qquad G(R)\geqq(2\delta)^{-1}L(1+\delta,\chi_1)+O(R^{-1/2+\varepsilon}q^{1/4+\varepsilon}),$
$L(1+\delta,\chi_1)\gg\delta.$

Lemma 5. *Let* $M<N$ *and let* $0\leqq\mathrm{Re}\,(s)\leqq 1/2$. *Then we have*

$$\begin{aligned}H(s,\chi)&=\sum_{n=1}^{\infty}\chi(n)B(n)\Big(\sum_{d|n}\theta_d\Big)^2 n^{-1-s}(e^{-n/N}-e^{-n/M})\\&=E(\chi)q^{-1}\varphi(q)L(1+\delta,\chi_1)G_q(R)^{-1}(M^{-s}-N^{-s})\Gamma(-s)\\&\quad+O(R^{2+\varepsilon}q(|s|+1)M^{-1+\varepsilon}),\end{aligned}$$

where $E(\chi)=1$ *if* χ *is principal and* $E(\chi)=0$ *otherwise and* $(M^{-s}-N^{-s})$ $\Gamma(-s)$ *is defined to be* $\log(N/M)$ *if* $s=0$.

Lemmas 1 and 2 are elementary. Lemma 3 is a special case of the Selberg sieve. Lemma 4 can easily be proved by observing the expression

$$\sum_{r=1}^{\infty}\mu^2(r)g(r)r^{-s}=\zeta(s+1)L(s+1+\delta,\chi_1)A(s),$$

where $A(s)$ is bounded for $\mathrm{Re}\,(s)>-1$ and $A(0)=1$. As for Lemma 5 we note that, by Lemmas 1 and 2, $H(s,\chi)$ is a difference of two Mellin transforms of $L(s,\chi)L(s+\delta,\chi\chi_1)F(s,\chi;\theta\circ\theta)$ and that, if χ is principal, $F(1,\chi;\theta\circ\theta)=G_q(R)^{-1}$ by Lemma 3.

3. Now we proceed as follows. By a familiar argument, it is sufficient to consider the set $\{(\rho_j,\chi^{(j)})\}$, $j\leqq J$, such that $L(\rho_j,\chi^{(j)})L(\rho_j+\delta,\chi^{(j)}\chi_1)=0$, $\mathrm{Re}\,(\rho_j)\geqq\alpha$, $|\mathrm{Im}\,(\rho_j)|\leqq T$, and $(\rho_j,\chi^{(j)})$ are $(\log qT)^{-1}$ well-spaced. Then, taking a Mellin transform of $L(s,\chi)L(s+\delta,\chi\chi_1)F(s,\chi;\xi\circ\theta)$, where ξ is defined in Lemma 4 of [6], we get, by Lemmas 1 and 2,

$$1\ll\Big|\sum_{z\leqq n\leqq Y^{1+\varepsilon}}\chi^{(j)}(n)B(n)\Big(\sum_{f|n}\xi_f\Big)\Big(\sum_{d|n}\theta_d\Big)n^{-\rho_j}e^{-n/Y}\Big|.$$

Here we have to assume $Y^\alpha\geqq(qTRz)^{1+\varepsilon}$. And then, by the Halász inequality [5; Lemma 1.7], we have

$$J^2\ll\sum_{n=1}^{\infty}B(n)\Big(\sum_{f|n}\xi_f\Big)^2 n^{1-2\alpha}e^{-n/Y}\sum_{j,k\leqq J}|H(\bar\rho_j+\rho_k-2\alpha,\overline{\chi^{(j)}}\chi^{(k)})|,$$

where $H(s,\chi)$ of Lemma 5 is used with $M=z^{1-\varepsilon}$, $N=Y^{1+2\varepsilon}$. According to Lemma 4 of [6], the first sum is $O(Y^{2(1-\alpha)})$, since $B(n)\leqq\tau(n)$ and $n^{1-2\alpha}e^{-n/Y}\ll Y^{2(1-\alpha)}n^{-\kappa}$, $\kappa=1+(\log Y)^{-1}$. As for the second sum we see, by Lemma 5, that it is

$$\ll Jq^{-1}\varphi(q)L(1+\delta,\chi_1)G_q(R)^{-1}\log Y+J^2R^2qTz^{-1+2\varepsilon}.$$

Now we set $R=q^{1/2+4\varepsilon}$, $z^{1-3\varepsilon}=qTR^2Y^{2(1-\alpha)}$, and thus $Y^{3\alpha-2}=(R^3q^2T^2)^{1+\varepsilon}$. Then, by Lemma 4, we find

$$J \ll Y^{2(1-\alpha)}\delta \log Y,$$

which ends our brief proof of the theorem.

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