1. *On the Deuring-Heilbronn Phenomenon. I*

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1. Let χ_1 be a real primitive character (mod q_1) and let suppose that $L(s,\chi_1)$ has an exceptional zero $1-\delta$, $0<\delta\ll(\log q_1)^{-1}$. Then Linnik [4] proved

Theorem. *There exist effective absolute-constants $c_1, c_2>0$ such that, if $\delta\log(qq_1(|\gamma|+2))\leqq c_1$, then every zero $\rho=\beta+i\gamma$ of $L(s,\chi)$ (mod q) satisfies*

$$\beta\leqq 1-c_2(\log(qq_1(|\gamma|+2)))^{-1}\log\left(\frac{ec_1}{\delta\log(qq_1(|\gamma|+2))}\right),$$

save for the case $\rho=1-\delta$, $\chi=\chi_1$.

Linnik's proof is quite difficult and elaborated. He used Brun's sieve as well as some convexity results on entire functions. Later Knapowski [3] simplified it considerably by appealing to the power sum method of Turán. And recently Montgomery [5] has found a quite simple version of Turán's method and applied it to deduce very elegantly the above theorem.

The purpose of the present note is to show very briefly that there is a conceptually much elementary and more direct way which depends only on Selberg's sieve and on some easy facts on $\zeta(s)$ and $L(s,\chi)$. Our argument can be considered to be a natural extension of Selberg's proof [5] (see also [1; p. 40] [2] [6]) of Linnik's zero-density theorem. The detailed account as well as some applications to the large sieve of the present method will appear elsewhere.

2. Before stating our main lemmas, we introduce the following notations:

$$B(n)=\sum_{d|n}\chi_1(d)d^{-\delta},\qquad g(r)=\prod_{p|r}\frac{(1+\chi_1(p)p^{-\delta}-\chi_1(p)p^{-1-\delta})}{(p-1)(1-\chi_1(p)p^{-1-\delta})},$$

$$\Psi_r(n)=\mu((n,r))g((n,r))^{-1},\qquad G(R)=\sum_{r\leqq R}\mu^2(r)g(r).$$

Then we have

Lemma 1. *If $\log N\gg\log(q_1R)$, then*

$$\sum_{N\leqq n<2N}B(n)\left\{\sum_{r\leqq R}\mu^2(r)g(r)\Psi_r(n)\right\}^2\ll L(1+\delta,\chi_1)G(R)N\cdot$$

Lemma 2. *If $\log R\gg\log q_1$, then*

$$G(R)\gg\delta^{-1}L(1+\delta,\chi_1).$$

Lemma 3. *If r is square-free and $\xi_d=O(\mu^2(d))$, then we have, for*

$\sigma>1$,

$$\sum_{n=1}^{\infty} B(n)\Psi_r(n)\Big(\sum_{d|n}\xi_d\Big)\chi(n)n^{-s}=L(s,\chi)L(s+\delta,\chi\chi_1)g(r)^{-1}\Phi_r(s,\chi),$$

where

$$\Phi_r(s,\chi)=\sum_{d=1}^{\infty}\xi_d\chi(d)\mu((r,d))d^{-s}\prod_{\substack{p|d\\ p\nmid r}}\Big(1+\frac{\chi_1(p)}{p^{\delta}}-\frac{\chi\chi_1(p)}{p^{s+\delta}}\Big)$$
$$\times\prod_{\substack{p\nmid d\\ p|r}}\Big\{\Big(1-\frac{1}{p}\Big)^{-1}\Big(1-\frac{\chi_1(p)}{p^{1+\delta}}\Big)^{-1}\Big(1-\frac{\chi(p)}{p^{s}}\Big)\Big(1-\frac{\chi\chi_1(p)}{p^{s+\delta}}\Big)-1\Big\}.$$

Lemma 4. *Let ε be a positive constant and let*

$$\lambda_j(d)=\mu(d)\Big(\log\frac{z^{1+j\varepsilon}}{d}\Big)^2 \quad \text{if } d\leqq z^{1+j\varepsilon} \text{ and } =0 \text{ otherwise.}$$

Further put $\xi_d=1/2(\varepsilon\log z)^{-2}(\lambda_0(d)-2\lambda_1(d)+\lambda_2(d))$. Then $\xi_d=\mu(d)$ if $d\leqq z$. And we have

$$\sum_{n=1}^{\infty}\tau(n)\Big(\sum_{d|n}\xi_d\Big)^2 n^{-\kappa}\ll 1,$$

where $\tau(n)$ is the divisor function and $\kappa=1+O((\log z)^{-1})$.

3. Now we argue as follows: If $\rho=\beta+i\gamma$, $\beta\geqq 3/4$, is a zero of $L(s,\chi)$, then we have, by Lemma 3 (but with ξ_d of Lemma 4),

$$\sum_{n\geqq z} B(n)\Psi_r(n)\Big(\sum_{d|n}\xi_d\Big)\chi(n)n^{-\rho}e^{-n/Y}=-1+O(Y^{-1/8}),$$

where we have to assume that $\log Y\gg\log(Rzqq_1(|\gamma|+2))$. Multiplying by $\mu^2(r)g(r)$ both sides and summing over $r\leqq R$, we get

$$\frac{1}{4}G(R)^2\leqq\sum_{n\geqq z}B(n)\Big\{\sum_{r\leqq R}\mu^2(r)g(r)\Psi_r(n)\Big\}^2 n^{-1}e^{-n/Y}$$
$$\times\sum_{n=1}^{\infty}\tau(n)\Big(\sum_{d|n}\xi_d\Big)^2 n^{1-2\beta}e^{-n/Y},$$

since $B(n)\leqq\tau(n)$. But we have $n^{1-2\beta}e^{-n/Y}\ll Y^{2(1-\beta)}n^{-\kappa'}$, where $\kappa'=1+(\log Y)^{-1}$, and so by Lemma 4 we see that, if $\log z\ll\log Y$, the last sum is $O(Y^{2(1-\beta)})$. On the other hand, if $\log z\gg\log(q_1R)$, we see, by Lemma 1, the first sum is $O(L(1+\delta,\chi_1)G(R)\log Y)$. Hence we have

$$Y^{2(1-\beta)}L(1+\delta,\chi_1)\log Y\gg G(R).$$

And by Lemma 2 we get the result of Linnik.

References

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