133. *On the Theory of Schlicht Functions.*

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1. Let the function

(1) $$f(z)=z+a_2z^2+\cdots\cdots$$

be regular and schlicht in $|z|<1$. It is well known that $|a_n|<en$. Recently Littlewood and Paley[1] show that the coefficients a_n are uniformly bounded, if $f(z)$ be odd. We can, however, proceed one step further, and prove that

(2) $$|a_n|<e^2,$$

for the odd schlicht function.

If the function (1) satisfies the relation

(3) $$f(\omega z)=\omega f(z),$$

where ω denotes any root of the equation $x^k=1$, k being a positive integer, we say, for convenience, that $f(z)$ is a function of the class k. Using this definition, the results stated above may be expressed as follows: if $f(z)$ is a function of the class k, then, for $k=1, 2$, we have

(4) $$n^{\frac{k-2}{k}}|a_n|<A.$$

The question arises whether the inequality (4) holds good also for $k>2$. The answer is affirmative in the case $k=3$, and the relation

(5) $$\sqrt[3]{n}\,|a_n|<e^3$$

can be demonstrated. Thus we can state the following theorem:

If $f(z)=z+\sum_2^\infty a_nz^n$ *is a function of the class* k $(k=1, 2, 3)$, *then*

(6) $$n^{\frac{k-2}{k}}|a_n|<e^k.$$

In general, if the function $zf'(z)$ is p-valent, i.e. the equation

$$zf'(z)=a$$

has at most p solutions in $|z|<1$, for every a, we can establish the inequality

(7) $$n^{\frac{k-2}{k}}|a_n|<p\,e^{\frac{1}{k}},$$

1) J. E. Littlewood and R. E. A. C. Paley: Journal of London Math. Soc. **7** (1933). See also E. Landau: Über ungerade schlichte Funktionen, Math. Zeits. **37** (1933).

where a_n denotes the coefficients of the function $f(z)$ of the class k. The index $(k-2)/k$ is the best possible one, as shown by the function

$$f(z)=\frac{z}{(1-z^k)^{2/k}}=z+\cdots\cdots.$$

2. To prove the relations (6) and (7), the following theorem is useful. If $f_k(z)$ is a function of the class k, then

$$(8)\qquad \frac{1}{2\pi}\int_0^{2\pi}|f_k(re^{i\theta})|^{\alpha p}d\theta\leqq A_p r^{\alpha p}(1-r^k)^{2\alpha(1-p)/k},$$

where $0\leqq r<1$, $0\leqq\alpha\leqq k/2$, $p>1$,

$$A_p=\text{Max}\,(1,\ p/(2p-2)).$$

Putting $k=1$, $\alpha=1/2$, $p=2\lambda$ $(2\lambda>1)$, (8) is reduced to an inequality of Littlewood.[1]

By means of (8), we can establish the relation

$$(9)\qquad \frac{1}{2\pi}\int_0^{2\pi}|f_k'(re^{i\theta})|^{\lambda}d\theta\leqq A(k,\lambda)(1+r^k)^{\lambda}(1-r^k)^{1-\lambda-2\lambda/k},$$

where $0\leqq r<1$, $2\lambda>k$, $A(k,\lambda)=\text{Max}\,(1,\ \lambda(2\lambda-k))$.

As further applications of (8), we can show that, if

$$f_k(z)=\sum a_n z^n\ \ (a_1=1),$$

then
$$\sum_{m=1}^{n}|a_m|^2<e^2 r^{4/k-1}\qquad (\text{for}\quad k=1,2,3),$$

$$\sum_{m=1}^{n}|a_m|^2<e\log 2n\qquad (\text{for}\quad k=4),$$

$$\sum_{m=1}^{\infty}|a_m|^2<k^2/(k-2)(k-4)\qquad (\text{for}\quad k>4).$$

3. The classification of the regular schlicht functions in the unit circle, in the above manner, can be employed to establish a theorem concerning schlicht character of a function. Let $f(z)$ be an analytic function. If $f(z)$ exists and does not vanish, then $f(z)$ is regular and schlicht in a circle C about z_0. The question arises how large the circle C can be taken in order to keep the above properties. Certainly, C cannot contain the greatest circle C' about z_0 in which $f'(z)$ exists and is different from zero. On the contrary, $f(z)$ may not be schlicht in C_0'. For instance, although the derivative of the function e^z does not vanish in the whole plane, yet e^z is not schlicht in any circle with the radius greater than π. We can, however, prove the following theorem: *if $f'(z)$ exists and does not vanish in the cirlce C with the center z_0, $f(z)$*

1) J. E. Littlewood: Proc. of London Math. Soc. (2), **23** (1925).

is schlicht in C, provided that

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<|f'(z_0)| \tag{10}$$

at every inner point z of C. Moreover, if the condition (10) is replaced by

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<M|f'(z_0)|, \tag{11}$$

where M denotes a constant >1.12, then the statement is false.

In fact, the function

$$f(z)=z+z^2+\frac{z^3}{3}$$

is not schlicht in $|z|<\sqrt{3}/2+\varepsilon=\rho$ $(\varepsilon>0,\ \rho<1)$, since

$$f\left(\frac{\sqrt{3}}{2}e^{5\pi i/6}\right)=f\left(\frac{\sqrt{3}}{2}e^{-5\pi i/6}\right).$$

But, when $|z|\leqq\rho$, we have

$$f'(z)=1+2z+z^2\neq 0,$$

$$\left|\frac{f(z)-f(0)}{z-0}-f'(0)\right|=\left|z+\frac{z^2}{3}\right|<\frac{\sqrt{3}}{2}+\frac{1}{4}+\varepsilon<1.12|f'(0)|,$$

if ε is sufficiently small.

It is probable that the multiplier M in (11), in order to keep the theorem well, cannot be greater than unity.