98. Connections in the Manifold Admitting Generalized Transformations.

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In the present paper a general manifold is defined in which to every point of a manifold X_n is associated a system of the quantities, $\overset{(1)}{P}{}^\nu_a, \overset{(2)}{P}{}^\nu_a, \ldots\ldots \overset{(h)}{P}{}^\nu_a$. We shall develop the notions of the point transformation for this general manifold, and then by an analogous method as in a previous paper[1] the connections will be established in it.

1. *The local geometry.* Consider an n dimensional space X_n of coordinates x^ν ($\nu=a_1, \ldots\ldots a_n$), and to each point in X_n corresponds a system of h mutual independent quantities $\overset{(1)}{P}{}^\nu_a, \ldots\ldots \overset{(h)}{P}{}^\nu_a$, whose directions are indeterminate, and $a=1, 2, \ldots\ldots K$. We consider $\overset{(1)}{P}{}^\nu_a$ as the elements of *K-spread*,[2] depending analytically on a system of parameters (u^a; $a=1, 2, \ldots\ldots K$). This new manifold is called *the general manifold.*

We shall now assume for the quantities $\overset{(1)}{P}{}^\nu_a, \ldots\ldots \overset{(h)}{P}{}^\nu_a$:

$$(1.1)\qquad d\overset{(i)}{P}{}^\nu_a = \overset{(i)}{\Psi}{}^\nu_{a/\lambda} dx^\lambda \qquad \begin{pmatrix} i=1, 2, \ldots\ldots h \\ a=1, 2, \ldots\ldots K \end{pmatrix},$$

where $\overset{(i)}{\Psi}{}^\nu_{a/\lambda}$ are arbitrary functions.

Let us consider the transformations

$$(1.2)\qquad 'x^\nu = 'x^\nu\,(x^\nu, \overset{(1)}{P}{}^\nu_a, \ldots\ldots \overset{(h)}{P}{}^\nu_a), \qquad \nu=a_1, \ldots\ldots a_n,$$

in the general manifold. By differentiation of (1.2), we get

$$(1.3)\qquad d'x^\nu = \left(\frac{\partial' x^\nu}{\partial x^\lambda} + \frac{\partial' x^\nu}{\partial \overset{(i)}{P}{}^\mu_a} \overset{(i)}{\Psi}{}^\mu_{a/\lambda}\right) dx^\lambda .$$

We make use the usual convention for indices about every one of the letters λ, i and a.

Any set of n quantities $V^\nu(x^\nu, \overset{(1)}{P}{}^\nu_a, \ldots\ldots \overset{(1)}{P}{}^\nu_a)$, ($\nu=a_1, \ldots\ldots a_n$), transformed by the transformations (1.2) into new n quantities $'V^\nu\,('x^\nu, '\overset{(1)}{P}{}^\nu_a, \ldots\ldots '\overset{(h)}{P}{}^\nu_a)$ in such a way that

1) T. Hosokawa: Science Reports, Tohoku Imp. University, **19** (1930), p. 37–51.
2) J. Douglas: Math. Annalen, **105** (1931), p. 707.

(1.4) $$'V^\nu=\left(\frac{\partial' x^\nu}{\partial x^\lambda}+\frac{\partial' x^\nu}{\partial \overset{(i)}{P}{}^\mu_a}\overset{(i)}{\Psi}{}^\mu_{a/\lambda}\right)V^\lambda,$$

will be called *a contravariant vector*, where $'\overset{(i)}{P}{}^\nu_a$ are the quantities at the point $'x^\nu$. A *covariant vector* is a set of n quantities W_λ which are transformed by (1.2) into

(1.5) $$'W_\mu=\left(\frac{\partial x^\lambda}{\partial' x^\mu}+\frac{\partial x^\lambda}{\partial' \overset{(i)}{P}{}^\omega_a}\,'\overset{(i)}{\Psi}{}^\omega_{a/\mu}\right)W_\lambda.$$

Let us now assume that the following relations are satisfied:

(1.6) $$\frac{\partial' x^\mu}{\partial \overset{(j)}{P}{}^\omega_b}\frac{\partial x^\lambda}{\partial' \overset{(i)}{P}{}^\kappa_a}\,'\overset{(i)}{\Psi}{}^\kappa_{a/\mu}\overset{(j)}{\Psi}{}^\omega_{b/\lambda}=0.$$

A tensor of the higher order is defined by the following:

$$'V^{\nu_1\cdots\nu_m}_{\lambda_1\cdots\lambda_n}=V^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n}\prod_{k=1}^{m}\left(\frac{\partial' x^{\nu_k}}{\partial x^{\alpha_k}}+\frac{\partial' x^{\nu_k}}{\partial \overset{(j)}{P}{}^\mu_a}\overset{(j)}{\Psi}{}^\mu_{a/\alpha_k}\right)\prod_{h=1}^{n}\left(\frac{\partial x^{\beta_h}}{\partial' x^{\lambda_h}}+\frac{\partial x^{\beta_h}}{\partial' \overset{(i)}{P}{}^\omega_b}\,'\overset{(i)}{\Psi}{}^\omega_{b/\lambda_h}\right).$$

2. *Linear connections.* We will define the connections of the contravariant and covariant vectors by the following equations:

(2.1) $$\nabla_\mu V^\nu=\frac{\partial V^\nu}{\partial x^\mu}+\frac{\partial V^\nu}{\partial \overset{(i)}{P}{}^\lambda_a}\overset{(i)}{\Psi}{}^\lambda_{a/\mu}+\Gamma^\nu_{\lambda\mu}V^\lambda$$

and

(2.2) $$\nabla_\mu W_\lambda=\frac{\partial W_\lambda}{\partial x^\mu}+\frac{\partial W_\lambda}{\partial \overset{(i)}{P}{}^\lambda_a}\overset{(i)}{\Psi}{}^\lambda_{a/\mu}-\Gamma'^\nu_{\lambda\mu}W_\nu.$$

The covariant derivatives $\nabla_\mu V^\nu$ are the components of a mixed tensor of the second order. Hence from the transformation (1.2), it is evident that if $\Gamma^\lambda_{\omega\mu}$ are functions of x^ν as well as $\overset{(i)}{P}{}^\nu_a$, and $'\Gamma^\nu_{\mu\alpha}$ of $'x^\nu$ as well as $'\overset{(i)}{P}{}^\nu_a$, then they must satisfy the equations

$$\frac{\partial^2{}' x^\nu}{\partial x^\mu\partial x^\omega}+\frac{\partial^2{}' x^\nu}{\partial \overset{(i)}{P}{}^\kappa_a\partial x^\omega}\overset{(i)}{\Psi}{}^\kappa_{a/\mu}+\frac{\partial^2{}' x^\nu}{\partial x^\mu\partial \overset{(i)}{P}{}^\sigma_a}\overset{(i)}{\Psi}{}^\sigma_{a/\omega}+\frac{\partial' x^\nu}{\partial \overset{(i)}{P}{}^\sigma_a}\frac{\partial \overset{(i)}{\Psi}{}^\sigma_{a/\omega}}{\partial x^\mu}+\frac{\partial^2{}' x^\nu}{\partial \overset{(j)}{P}{}^\sigma_a\partial \overset{(i)}{P}{}^\rho_b}\overset{(j)}{\Psi}{}^\sigma_{a/\mu}\overset{(i)}{\Psi}{}^\rho_{b/\omega}$$

$$+\frac{\partial' x^\nu}{\partial \overset{(i)}{P}{}^\rho_a}\frac{\partial \overset{(i)}{\Psi}{}^\rho_{a/\omega}}{\partial \overset{(j)}{P}{}^\sigma_b}\overset{(j)}{\Psi}{}^\sigma_{b/\mu}+{}'\Gamma^\nu_{\beta\alpha}\left(\frac{\partial' x^\beta}{\partial x^\omega}+\frac{\partial' x^\beta}{\partial \overset{(i)}{P}{}^\sigma_a}\overset{(i)}{\Psi}{}^\sigma_{a/\omega}\right)\left(\frac{\partial' x^\alpha}{\partial x^\mu}+\frac{\partial' x^\alpha}{\partial \overset{(j)}{P}{}^\tau_a}\overset{(j)}{\Psi}{}^\tau_{a/\mu}\right)$$

$$=\Gamma^\lambda_{\omega\mu}\left(\frac{\partial' x^\nu}{\partial x^\lambda}+\frac{\partial' x^\nu}{\partial \overset{(j)}{P}{}^\sigma_a}\overset{(j)}{\Psi}{}^\sigma_{a/\lambda}\right).$$

In the same manner as that of the general linear displacements,[1] we get

$$\Gamma^{\nu}_{\lambda\mu}=\left\{{\lambda\mu \atop \nu}\right\}+T_{\lambda\mu}^{\cdot\cdot\nu}+\overline{\left\{{\lambda\mu \atop \nu}\right\}}, \qquad \Gamma'^{\nu}_{\lambda\mu}=\left\{{\lambda\mu \atop \nu}\right\}+T'^{\cdot\cdot\nu}_{\lambda\mu}+\overline{\left\{{\lambda\mu \atop \nu}\right\}}, \tag{2.3}$$

where

$$\overline{\left\{{\lambda\mu \atop \nu}\right\}}=\frac{1}{2}g^{\nu\omega}\left(\frac{\partial g_{\lambda\omega}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}\overset{(i)}{\Psi}{}^{\tau}_{a/\mu}+\frac{\partial g_{\omega\mu}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}\overset{(i)}{\Psi}{}^{\tau}_{a/\lambda}-\frac{\partial g_{\lambda\mu}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}\overset{(i)}{\Psi}{}^{\tau}_{a/\omega}\right). \tag{2.4}$$

3. *Curvature tensors.* From (2.1) and (2.2) we have

$$\nabla_{[\mu}\nabla_{\nu]}V^{\lambda}=-\frac{1}{2}R_{\nu\mu\rho}^{\cdot\cdot\cdot\lambda}V^{\rho}+\frac{1}{2}\overset{(i)}{K}{}_{a/\nu\mu}^{\cdot\cdot\tau}\frac{\partial V^{\lambda}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}+S'^{\cdot\cdot\alpha}_{\mu\nu}\nabla_{\alpha}V^{\lambda},$$

where

$$R_{\nu\mu\rho}^{\cdot\cdot\cdot\lambda}=\frac{\partial\Gamma^{\lambda}_{\rho\nu}}{\partial x^{\mu}}-\frac{\partial\Gamma^{\lambda}_{\rho\mu}}{\partial x^{\nu}}+\Gamma^{\lambda}_{\omega\mu}\Gamma^{\omega}_{\rho\nu}-\Gamma^{\lambda}_{\omega\nu}\Gamma^{\omega}_{\rho\mu}+\frac{\partial\Gamma^{\lambda}_{\rho\nu}}{\partial \overset{(i)}{P}{}^{\omega}_{a}}\overset{(i)}{\Psi}{}^{\omega}_{a/\mu}-\frac{\partial\Gamma^{\lambda}_{\rho\mu}}{\partial \overset{(i)}{P}{}^{\omega}_{a}}\overset{(i)}{\Psi}{}^{\omega}_{a/\nu},$$

and

$$\overset{(i)}{K}{}_{a/\nu\mu}^{\cdot\cdot\lambda}=\frac{\partial\overset{(i)}{\Psi}{}^{\tau}_{a/\nu}}{\partial x^{\mu}}-\frac{\partial\overset{(i)}{\Psi}{}^{\tau}_{a/\mu}}{\partial x^{\nu}}+\frac{\partial\overset{(i)}{\Psi}{}^{\tau}_{a/\nu}}{\partial \overset{(j)}{P}{}^{\omega}_{b}}\overset{(j)}{\Psi}{}^{\omega}_{b/\mu}-\frac{\partial\overset{(i)}{\Psi}{}^{\tau}_{a/\mu}}{\partial \overset{(j)}{P}{}^{\omega}_{b}}\overset{(j)}{\Psi}{}^{\omega}_{b/\nu}.$$

Similarly, we obtain

$$R'^{\cdot\cdot\cdot\lambda}_{\nu\mu\rho}=\frac{\partial\Gamma'^{\lambda}_{\rho\nu}}{\partial x^{\mu}}-\frac{\partial\Gamma'^{\lambda}_{\rho\mu}}{\partial x^{\nu}}+\Gamma'^{\lambda}_{\omega\mu}\Gamma'^{\omega}_{\rho\nu}-\Gamma'^{\lambda}_{\omega\nu}\Gamma'^{\omega}_{\rho\mu}+\frac{\partial\Gamma'^{\lambda}_{\rho\nu}}{\partial \overset{(i)}{P}{}^{\omega}_{a}}\overset{(i)}{\Psi}{}^{\omega}_{a/\mu}-\frac{\partial\Gamma'^{\lambda}_{\rho\mu}}{\partial \overset{(i)}{P}{}^{\omega}_{a}}\overset{(i)}{\Psi}{}^{\omega}_{a/\nu},$$

but

$$\nabla_{[\mu}\nabla_{\nu]}W_{\rho}=-\frac{1}{2}R'^{\cdot\cdot\cdot\lambda}_{\nu\mu\rho}W_{\lambda}+\frac{1}{2}\overset{(i)}{K}{}_{a/\nu\mu}^{\cdot\cdot\tau}\frac{\partial W_{\rho}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}+S'^{\cdot\cdot\alpha}_{\mu\nu}\nabla_{\alpha}W_{\rho}. \tag{3.1}$$

We will call $R_{\nu\mu\rho}^{\cdot\cdot\cdot\lambda}$ and $R'^{\cdot\cdot\cdot\lambda}_{\nu\mu\rho}$ the *curvature tensors.*

From the formula: $\nabla_{[\omega}\nabla_{\mu]}(\Psi\Phi)=\Psi\nabla_{[\omega}\nabla_{\mu]}\Phi+\Phi\nabla_{[\omega}\nabla_{\mu]}\Psi$, we have

$$2\nabla_{[\xi}\nabla_{\omega]}\nabla_{\mu}W_{\lambda}=-R'^{\cdot\cdot\cdot\alpha}_{\omega\xi\mu}\nabla_{\alpha}W_{\lambda}-R'^{\cdot\cdot\cdot\alpha}_{\omega\xi\lambda}\nabla_{\mu}W_{\alpha}+\overset{(i)}{K}{}_{a/\omega\xi}^{\cdot\cdot\tau}\frac{\partial(\nabla_{\mu}W_{\lambda})}{\partial \overset{(i)}{P}{}^{\tau}_{a}}+2S'^{\cdot\cdot\tau}_{\xi\omega}\nabla_{\tau}\nabla_{\mu}W_{\lambda}. \tag{3.2}$$

From (3.1) it follows that

$$2\nabla_{\xi}\nabla_{[\omega}\nabla_{\mu]}W_{\lambda}=\nabla_{\xi}\left(-R'^{\cdot\cdot\cdot\alpha}_{\mu\omega\lambda}W_{\alpha}+\overset{(i)}{K}{}_{a/\mu\omega}^{\cdot\cdot\tau}\frac{\partial W_{\lambda}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}+2S'^{\cdot\cdot\alpha}_{\omega\mu}\nabla_{\alpha}W_{\lambda}\right). \tag{3.3}$$

From (3.2) and (3.3) we have the following identities:

$$\begin{aligned}\nabla_{[\xi}\Big(-R'^{\cdot\cdot\cdot\alpha}_{\mu\omega]\lambda}W_{\alpha}&+\overset{(i)}{K}{}_{a/\mu\omega]}^{\cdot\cdot\tau}\frac{\partial W_{\lambda}}{\partial \overset{(i)}{P}{}^{\tau}_{a}}+2S'^{\cdot\cdot\alpha}_{\omega\mu]}\nabla_{\alpha}W_{\lambda}\Big)+R'^{\cdot\cdot\cdot\alpha}_{[\omega\xi\mu]}\nabla_{\alpha}W_{\lambda}\\&+R'^{\cdot\cdot\cdot\alpha}_{[\omega\xi|\lambda|}\nabla_{\mu]}W_{\alpha}-\overset{(i)}{K}{}_{a/[\omega\xi}^{\cdot\cdot\tau}\frac{\partial(\nabla_{\mu]}W_{\lambda})}{\partial \overset{(i)}{P}{}^{\tau}_{a}}-2S'^{\cdot\cdot\tau}_{[\xi\omega}\nabla_{|\tau|}\nabla_{\mu]}W_{\lambda}=0.\end{aligned} \tag{3.4}$$

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1) T. Hosokawa: loc. cit., p. 40.

In consequence of these identities we have

$$(3.5)\qquad -\nabla_{[\xi}R'_{\mu\omega]\lambda}{}^{\alpha}+C_{[\xi|\beta|}{}^{\alpha}R'_{\omega\mu]\lambda}{}^{\beta}+2R'_{\beta[\xi|\lambda|}{}^{\alpha}S'_{\mu\omega]}{}^{\beta}-\overset{(i)}{K}_{a/[\xi\mu}{}^{\tau}\frac{\partial\Gamma'_{|\lambda|\omega]}}{\partial\overset{(i)}{P}{}^{\tau}_{a}}=0,$$

(3.6)

$$\nabla_{[\xi}\overset{(i)}{K}_{|a|\mu\omega]}{}^{\tau}-C_{[\xi|\beta|}{}^{\tau}\overset{(i)}{K}_{|a|/\mu\omega}{}^{\beta}+\Gamma'^{\tau}_{\beta[\xi}\overset{(i)}{K}_{|a|/\mu\omega}{}^{\beta}+2\overset{(i)}{K}_{a/\alpha[\xi}{}^{\tau}S'_{\omega\mu]}{}^{\alpha}-\frac{\partial\overset{(i)}{\Psi}{}^{\tau}_{a/[\xi}}{\partial\overset{(j)}{P}{}^{\beta}_{b}}\overset{(i)}{K}_{|b|\mu\omega]}{}^{\beta}=0$$

and

$$(3.7)\qquad R'_{[\xi\mu\omega]}{}^{\alpha}+4S'_{[\mu\xi}{}^{\varkappa}S'_{\omega]\varkappa}{}^{\alpha}+2\nabla_{[\xi}S'_{\omega\mu]}{}^{\alpha}-2S'_{[\mu\xi}{}^{\beta}C_{\omega]\beta}{}^{\alpha}=0\,.$$

The relations (3.5) correspond to the identities of *Bianchi.*

In the equations (1.1) and (1.2) put respectively

$$\overset{(k+1)}{P}{}^{\nu}_{a}=\delta\overset{(k)}{P}{}^{\nu}_{a}=d\overset{(k)}{P}{}^{\nu}_{a}+\overset{(k)}{\Gamma}{}^{\nu}_{\lambda\mu}\overset{(k)}{P}{}^{\lambda}_{a}dx^{\mu}$$

and $$'x^{\nu}='x^{\nu}(x)\qquad(\nu=a_1,\ \ldots\ldots\ a_n)\,,$$

moreover put $K=1$, then we obtain the case, studied by *A. Kawaguchi.*[1)]

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1) A. Kawaguchi: Proc. 7 (1931), 211–214.