40. *On the Distribution of Zero Points of the Derivatives of an Integral Transcendental Function of Order* $\rho \leqq 1$.

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1. Recently I have proved the following theorem which is a modified form of a theorem enunciated by Mr. Takahashi:[1]

THEOREM I.[2] *Let* $\{g_n(z)\}$ *be a sequence of functions satisfying the following conditions:*

(i) $g_n(z)$ *is regular and analytic for* $|z| \leqq R$;

(ii) $g_n(z) = z^n\{1 + h_n(z)\}$, *where* $h_n(z)$ *is regular and analytic for* $|z| \leqq R$ *and vanishes at the origin;*

(iii) *there exists a finite constant* λ *for which*

$$\overline{\lim_{n=\infty}} |h_n(z)| \leqq \lambda \qquad \text{for} \qquad |z| \leqq R.$$

Then any function $f(z)$ *regular and analytic for* $|z| \leqq r$ *can be expanded in one and only one way into the series of the form*

$$f(z) = \sum_{n=0}^{\infty} c_n g_n(z),$$

which converges absolutely and uniformly for

$$|z| \leqq r_0 < \min\left(r, \frac{R}{1+\lambda}\right).$$

2. Let us consider a set $\{\alpha_n\}$ of points such that

$$\overline{\lim_{n=\infty}} |\alpha_n| = L$$

and put

$$g_n(z) = z^n e^{\bar{\alpha}_n z} = z^n\{1 + h_n(z)\}, \quad (|z| \leqq R, \ n = 0, 1, 2, \dots\dots).$$

Since $g_n(z)$ is regular and analytic for any finite value of R, and moreover

$$|h_n(z)| \leqq e^{|\alpha_n| R} - 1 \qquad \text{for} \qquad |z| \leqq R,$$

$$\overline{\lim_{n=\infty}} |h_n(z)| = e^{LR} - 1 (= \lambda \text{ say}) \qquad \text{for} \qquad |z| \leqq R,$$

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1) S. Takahashi: A remark on Mr. Widder's theorem, Tohoku Math. Journal, **33** (1930), 48.

2) The proof of this theorem will be given in my paper "On the expansion of analytic functions in a series of analytic functions and its applications etc." which will appear in Proc. Phys.-Math. Soc. of Japan.

the function $h_n(z)$ satisfies the conditions in Theorem I. On the other hand, we can easily show that the maximum of Re^{-LR} takes the value $\frac{1}{Le}$ given by $R=\frac{1}{L}$, so that we can state the following

THEOREM II. *Let $\{\alpha_n\}$ be a set of points such that*

$$\overline{\lim_{n=\infty}}|\alpha_n|=L<\infty .$$

Then any function $\phi(z)$ regular and analytic for $|z|\leqq r$ can be expanded in one and only one way into the series of the form

$$\phi(z)=\sum_{n=0}^{\infty} c_n z^n e^{\bar{\alpha}_n z},$$

which converges absolutely and uniformly for $|z|\leqq r_0<\min\left(r, \frac{1}{Le}\right)$.

3. Now let $f(z)$ be an integral transcendental function of type $\sigma(<1)$, and of the first order, and write

$$f(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n .$$

Then by the use of Stirling's formula, we can easily show that the function $\phi(z)$ defined by

$$\phi(z)=\sum_{n=0}^{\infty} a_n z^n$$

is regular and analytic for $|z|<\frac{1}{\sigma}$, so that it can be expanded uniquely into the series of the form

$$\text{(A)} \qquad \phi(z)=\sum_{n=0}^{\infty} c_n z^n e^{\bar{\alpha}_n z}, \qquad (\overline{\lim_{n=\infty}}|\alpha_n|=L)$$

which converges absolutely and uniformly for $|z|\leqq r<\min\left(\frac{1}{\sigma}, \frac{1}{Le}\right)$.

If we assume that $L<\frac{1}{e}$, it is obvious that $1<\min\left(\frac{1}{\sigma}, \frac{1}{Le}\right)$, whence, in this case, the series (A) converges absolutely and uniformly for $|z|\leqq 1$.

Taking the conjugate values of both sides of (A), multiplying by $\phi(z)$ and integrating term by term, we obtain

$$\frac{1}{2\pi}\int_{|z|=1}|\phi(z)|^2|dz|=\sum_{n=0}^{\infty}\bar{c}_n\frac{1}{2\pi}\int_{|z|=1}\phi(z)\bar{z}^n e^{\alpha_n \bar{z}}|dz| .$$

On the other hand we have

$$f(x)=\frac{1}{2\pi i}\int_{|z|=1}\phi(z)\frac{e^{\frac{x}{z}}}{z}dz=\frac{1}{2\pi}\int_{|z|=1}\phi(z)e^{x\bar{z}}\,|dz|\,,$$

so that

$$f^{(n)}(a_n)=\frac{1}{2\pi}\int_{|z|=1}\phi(z)\bar{z}^n e^{a_n\bar{z}}\,|dz|\,,\qquad (n=0,\ 1,\ 2,\ \ldots\ldots),$$

from which it follows that

$$\frac{1}{2\pi}\int_{|z|=1}|\phi(z)|^2\,|dz|=\sum_{n=0}^{\infty}\bar{c}_n f^{(n)}(a_n)\,.$$

From this equality we obtain the

THEOREM III. *Let $f(z)$ be an integral transcendental function of type $\sigma(<1)$, order 1, and let a_n be a zero of $f^{(n)}(z)$.*

If

$$\overline{\lim_{n=\infty}}\,|a_n|=L<\frac{1}{e}\,,$$

$f(z)$ should vanish identically.

4. We are now in a position to prove the following

THEOREM IV. *Let $f(z)$ be an integral transcendental function of type σ, order 1, and let a_n be a zero of $f^{(n)}(z)$.*

If

$$\overline{\lim_{n=\infty}}\,|a_n-z_0|=L<\frac{1}{\sigma e}\,,$$

$f(z)$ should vanish identically, where z_0 is a fixed point.[1]

PROOF. Without any loss of generality we can put $z_0=0$.

Let δ be an arbitrary positive constant and put

$$f^*(z)=f\left(\frac{z}{\sigma+\delta}\right)$$

and

$$x_n=(\sigma+\delta)a_n\,,\qquad (n=0,\ 1,\ 2,\ \ldots\ldots).$$

Then $f^*(z)$ is an integral transcendental function of type $\sigma'=\frac{\sigma}{\sigma+\delta}(<1)$, and of the first order; moreover x_n is a zero of $f^{*(n)}(z)$ with the condition

$$\overline{\lim_{n=\infty}}\,|x_n|=(\sigma+\delta)L\,.$$

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1) In my paper loc. cit., I have generalized this theorem for the case where the order ρ is any finite positive constant.

Therefore from Theorem III, $f^*(z)$ and accordingly $f(z)$ should vanish identically provided that

$$(\sigma+\delta)L<\frac{1}{e} \quad \text{or} \quad L<\frac{1}{e(\sigma+\delta)}.$$

Since δ is arbitrary, our theorem has been completely established.

Particularly if we put $\sigma=0$, we get

THEOREM V. *Let $f(z)$ be an integral transcendental function of order $\rho(<1)$ or of minimal type, and of the first order, and let a_n be a zero of $f^{(n)}(z)$.*

Then it must be

$$\overline{\lim_{n=\infty}}|a_n|=\infty,$$

and if this is not the case, $f(z)$ should vanish identically.