117. On the Expansion of Analytic Function.

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Prof. G. Julia has treated in his recent paper[1] the convergency of the series of the form $\sum_{n=0}^{\infty} a_n f(z^n)$, and the representation of any analytic function regular in the vicinity of the origin, by the series of the form above stated, where $f(z)$ is a polynomial or an entire function, such that

$$f(0)=0, \qquad f'(0)=1.$$

Suggested by this paper, I have obtained a very simple proof for Widder's theorem[2] on the expansion of an analytic function in generalized Taylor's series. This theorem may, somewhat modified from its original form, be stated as follows:

Theorem. Let $P_n(z)$ be the functions satisfying the following conditions in the domain $|z| \leqq R$:

(1) $P_n(z)$ are regular analytic in $|z| \leqq R$;

(2) $P_n(z)=1+h_n(z)$, where

$$h_n(z)=b_1^{(n)}z+b_2^{(n)}z^2+\cdots\cdots, \quad \text{for } |z| \leqq R, \quad (n=0, 1, 2, \ldots\ldots);$$

(3) $|h_n(z)| \leqq \dfrac{M}{n+1}$ for $|z| \leqq R$.

Then any analytic function $f(z)$, regular in $|z| \leqq \rho \leqq R$, can be expanded in one and only one way into a series of the form $\sum_{n=0}^{\infty} a_n z^n P_n(z)$, convergent in $|z| < \rho$. Moreover, the coefficients a_n are determined by the formula

$$(A) \qquad a_n=(-1)^n \begin{vmatrix} f(0) & 1 & & & & \\ \dfrac{f'(0)}{1!} & b_1^{(0)} & 1 & & & \\ \dfrac{f''(0)}{2!} & b_2^{(0)} & b_1^{(1)} & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{f^{(n)}(0)}{n!} & b_n^{(0)} & b_{n-1}^{(1)} & b_{n-2}^{(2)} & \cdots\cdots & b_1^{(n-1)} \end{vmatrix}.$$

1) Acta Mathematica, **54** (1930), 263-295.

2) Trans. Amer. Math. Soc., **31** (1929), 43-52. I have already published another simple proof of Widder's theorem. See Tôhoku Math. Journ., **33** (1930), 48-54.

Proof. Let $f(z)$ be an analytic function regular in $|z|\leqq\rho\leqq R$. We shall show that we can determine a_n such that the series $\sum_{n=0}^{\infty}a_nz^nP_n(z)$ converges uniformly for $|z|\leqq\rho'<\rho$ and represents the given function $f(z)$.

1. *Determination of a_n. Unicity of the expansion.*

If the preceding expansion is possible and has the required properties, then we must have

$$f(z)=\sum_{n=0}^{\infty}a_nz^nP_n(z)$$

in the domain, where $\sum_{n=0}^{\infty}a_nz^nP_n(z)$ is uniformly convergent.

Now let C be a circle with centre $z=0$, lying in the domain, where $\sum_{n=0}^{\infty}a_nz^nP_n(z)$ uniformly converges.

Then it is easily seen that

$$\int_C\frac{f(z)}{z^{n+1}}dz=\int_C\frac{a_0+a_1zP_1(z)+a_2z^2P_2(z)+\cdots\cdots+a_nz^nP_n(z)}{z^{n+1}}dz,$$

and since

$$\int_C\frac{a_nz^nP_n(z)}{z^{n+1}}dz=2\pi ia_n,$$

we have

$$\text{(B)}\quad a_n=\frac{1}{2\pi i}\int_C\frac{f(z)-a_0-a_1zP_1(z)-\cdots\cdots-a_{n-1}z^{n-1}P_{n-1}(z)}{z^{n+1}}dz.$$

By our hypothesis (2), the formula (B) becomes

$$\text{(C)}\quad a_n=\frac{f^{(n)}(0)}{n!}-a_1b_{n-1}^{(1)}-a_2b_{n-2}^{(2)}-\cdots\cdots-a_{n-1}b_1^{(n-1)}.$$

This recurring formula (C) gives us immediately the formula (A).

Thus all a_n are uniquely determined.

2. *Convergency of the expansion formed with the coefficients* (B).

We have by the hypothesis (3) the inequalities

$$|P_n(z)|\leqq1+\frac{M}{n+1},$$

$$\left|\frac{f(z)-a_0}{z}\right|<N$$

for $|z|\leqq\rho$. Now let the radius of the circle C be ρ. Then the formula (B) gives us the inequality

$$|a_n| < \rho^{-n}\left\{N\rho + |a_1|\rho\left(1+\frac{M}{2}\right) + |a_2|\rho^2\left(1+\frac{M}{3}\right) + \cdots\cdots \right.$$
$$\left. + |a_{n-1}|\rho^{n-1}\left(1+\frac{M}{n}\right)\right\},$$

which may be written

$$S_n - S_{n-1} < S_{n-1}\left(1+\frac{M}{n+1}\right) \quad \text{or} \quad S_n < S_{n-1}\left(2+\frac{M}{n+1}\right),$$

by putting

$$S_n = N\rho + |a_1|\rho\left(1+\frac{M}{2}\right) + |a_2|\rho^2\left(1+\frac{M}{3}\right) + \cdots\cdots$$
$$+ |a_n|\rho^n\left(1+\frac{M}{n+1}\right),$$

whence we get

$$S_n < S_1\left(2+\frac{M}{3}\right)\left(2+\frac{M}{4}\right)\cdots\cdots\left(2+\frac{M}{n+1}\right).$$

Now

$$S_1 = N\rho + |a_1|\rho\left(1+\frac{M}{2}\right),$$

and if we observe that

$$|a_1| = |f'(0)| = \left|\lim_{z\to 0}\frac{f(z)-a_0}{z}\right| < N,$$

we immediately have

$$S_1 < N\rho\left(2+\frac{M}{2}\right).$$

Therefore

$$S_n < N\rho\left(2+\frac{M}{2}\right)\left(2+\frac{M}{3}\right)\cdots\cdots\left(2+\frac{M}{n+1}\right),$$

and whence follows the inequality

$$|a_n| < N\rho^{-(n-1)}\left(2+\frac{M}{2}\right)\left(2+\frac{M}{3}\right)\cdots\cdots\left(2+\frac{M}{n}\right).$$

Thus we get

$$\sum_{n=1}^{\infty} N\rho^{-(n-1)}\left(2+\frac{M}{2}\right)\left(2+\frac{M}{3}\right)\cdots\cdots\left(2+\frac{M}{n}\right)\left(1+\frac{M}{n+1}\right)z^n$$

as the dominant series of $\sum_{n=1}^{\infty} a_n z^n P_n(z)$, and it is easily seen that this dominant series is absolutely and uniformly convergent in the domain $|z| \leqq \rho' < \rho$.

Therefore $\sum_{n=0}^{\infty} a_n z^n P_n(z)$ is uniformly convergent in $|z| \leqq \rho' < \rho$ and Widder's theorem is thus completely proved.

N. B. From our method of proof, it is not difficult to see that we can take, instead of (3), a more unrestricted condition

$$|h_n(z)| \leqq \frac{M}{\log(1+n)} \quad \text{for} \quad |z| \leqq R.$$